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比例代表制選挙におけるブロック別ドント式についての非確率論的及び確率論的考察

(Non-probabilistic and probabilistic approaches to the d'Hondt system of proportional representation with blocs)

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Abstract

It is usually believed in Japan that the d'Hondt system with blocs gives an advantage to large parties. If we regard the number of votes that each party gets as a constant (a non-probabilistic approach), not a random variable, this is not generally correct. I give an easy counter-example, also another by *falsifying* the actual data, giving more votes to the largest party, and fewer votes to other parties. In a non-probabilistic approach, I show some inequalities on the number of seats that each party wins. I give a rigorous statement and a proof that, under the d'Hondt system, a merger does not decrease seats unless losing support. If the proportions of the votes that a party gets are approximately independent of blocs, then the blocs give a disadvantage as long as it is a question whether the party wins a seat or not. If we regard the number of votes that each party gets as a random variable (a probabilistic approach), then the d'Hondt system with blocs gives an advantage to large parties in the sense of the expectation under some assumptions.

1. Introduction

On October 20, 1996, the election of the Lower House of the Japanese Diet was held for the first time under a new system, which was introduced in 1994. The new electoral system comprises 300 single-seat constituencies and 200 proportional representation (PR) seats by the d'Hondt system[1] with 11 blocs[2] (districts). The old one is the single nontransferable vote system in medium-sized districts, which is discussed by, e.g., Taagepera and Shugart (1989, p. 28) and Cox (1996). It is well known that the single-seat system gives a great advantage to the largest party, and the election result also proves this. In the following discussion, I consider mathematically whether the d'Hondt system of PR with blocs gives an advantage to large parties or not.

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[1] D'Hondt (1878, 1882) proposed his system in Belgium, and it was introduced in France in 1899. Hagenbach-Bischoff *et al.* (1884, pp. 26–27) gave a method of using r_* in Section 2. Hagenbach-Bischoff (1888) proposed an easier way of calculation to reach the same effect in Switzerland, which is called the Hagenbach-Bischoff system today. According to Fujita (1978, pp. 101–104), this was proposed in 1892. His contribution is important today, however, in theoretical sense, which I shall state below Theorem 1, rather than to have proposed an easier way of calculation. Using *Mathematica* for Macintosh, I calculated the numbers of seats by the method of d'Hondt in few seconds even if the magnitude is 200. See also Hagenbach-Bischoff (1908), Moriguchi (1925), Birke (1961), Mizuki (1967), and Rokkan (1968).

[2] In Japan, we use the loanword "bloc" for a constituency of the PR.

Yamamoto *et al.* (1996) and anonymous authors (1996b) consider whether each party would win more seats if the PR were carried out under a constituency covering the whole nation (i.e., not divided into blocs, "Nation" in tables). Their result is given in Table 1.

TABLE 1. PR seats

	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Actual	70	60	35	24	11	0	0	0	0	200
Nation	66	57	32	26	13	3	2	1	0	200
Increment	−4	−3	−3	+2	+2	+3	+2	+1	0	0

LDP	Liberal Democratic Party
NFP	Shinshinto (New Frontier Party)
MIN	Minshuto (Democratic Party of Japan)
JCP	Japanese Communist Party
SDP	Social Democratic Party
NSP	New Socialist Party
NPS	New Party Sakigake
JR	Liberal League (Jiyu Rengo)
DRL	Democratic Reform League

English names and their abbreviations are used according to annonymous authors (1996a).
For the detailed data, see Table 4 in Appendix C.

We see that the blocs gave an advantage to large parties and a disadvantage to small ones in this election. Yamamoto *et al.* (1996) point out, "Generally, the larger the magnitude, the smaller the percentage of votes it becomes to win a seat. To win a seat without fail, in the Kinki bloc, where the magnitude is 33, a party needs 2.9% of the votes; in the Shikoku bloc, where the magnitude is 7, it needs 12.5% of the votes; if the magnitude is 200, to get 0.5% of the votes is enough."[3] This is correct as will be seen later.

The anonymous authors (1996b) conclude, "The smaller the constituencies, the more advantageous it is to large parties. The larger the constituencies, the more advantageous it is to small parties and medium-sized ones."[3] Using data of other elections, Nisihira (1990, pp. 73–77) concludes, "Obviously the d'Hondt system of PR with blocs gives a great advantage to large parties."[3] A similar statement is found in Nisihira (1981, pp. 147–153), too. However, this does not generally hold. An easy counter-example is as follows:

Assume that there are 2 blocs $B^{(1)}$ and $B^{(2)}$, and 4 parties P_1, P_2, P_3, and P_4 run. We select 4 seats in each bloc. Then a counter-example is given in Table 2.

TABLE 2. Counter-example

	P_1	P_2	P_3	P_4	Total
Votes in $B^{(1)}$	9	8	7	5	29
Seats in $B^{(1)}$	1	1	1	1	4
Votes in $B^{(2)}$	9	8	7	5	29
Seats in $B^{(2)}$	1	1	1	1	4
Total seats	2	2	2	2	8
Total votes	18	16	14	10	58
Seats under Nation	3	2	2	1	8
Increment of seats	+1	0	0	−1	0

The reader might say, "This is an artificial counter-example since the numbers of votes are too small." However, we may multiply them by a positive constant, so this criticism does not make sense. It is easy to

[3] English translation by me.

make theoretic explanation of a disadvantage to the NSP, the NPS, and the JR, but it is not easy to do so to the JCP or the SDP.

Furthermore, we can make a counter-example by "*falsifying*" the actual data under the following restriction on the number of votes that each party gets:

LDP	(Actual)<(*Falsified*) in all blocs.
NFP, MIN, JCP, SDP	(Actual)$\geq$(*Falsified*) in all blocs.
NSP, NPS, JR	(Actual)>(*Falsified*) for the total numbers of votes with respect to the blocs.
DRL	It does not run for the *falsified* data.

The result is given in Table 3. Here I use italic numerals for the *falsified* data.

TABLE 3. PR seats based on *falsified* data

	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
With Blocs	*69*	*60*	*32*	*23*	*10*	*3*	*2*	*1*	*0*	*200*
Nation	*77*	*58*	*31*	*22*	*9*	*2*	*1*	*0*	*0*	*200*
Increment	*+8*	*−2*	*−1*	*−1*	*−1*	*−1*	*−1*	*−1*	*0*	*0*

For the detailed data, see Table 5 in Appendix C.

Of course, I have made the *falsified* data artificially. At present, it is difficult to know how I have done so because the reader does not know the meaning of θ_j nor "Estimates" yet.

2. A non-probabilistic approach: Part 1

First, I shall consider non-probabilistically in a fixed constituency that we select S seats, that is, S is a bloc (district) magnitude, where S is a given positive integer. I shall also apply the result and make numerical comparisons between the d'Hondt system with the blocs and the case of the constituency covering the whole nation. I shall use the following notation: n parties $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_j, \ldots, \mathrm{P}_n$ run and the party P_j gets v_j votes for $j = 1, \ldots, n$, where v_j is a nonnegative[4] integer.[5] In this section, I regard v_j as a constant, not a random variable. Denote $V := \sum_{j=1}^{n} v_j$, which is the total number of valid ballots, and assume that $V \neq 0$. Here I use a capital letter for a variable expressed as a total with respect to $j = 1, 2, \ldots, n$. Denote $p_j := v_j/V$, which is the relative proportion of the votes that the party P_j gets. Clearly $0 \leq p_j \leq 1$ and $\sum_{j=1}^{n} p_j = 1$ hold. The number of perfect PR seats of the party P_j is Sp_j, which is impossible to carry out except very special cases because it is not an integer. I shall consider the d'Hondt system of PR. Denote the number of seats that the party P_j wins by s_j, which is a nonnegative integer satisfying $\sum_{j=1}^{n} s_j = S$. (*Remark: This is not a definition of S, but S is an originally given constant.*) A definition[6] of $\{s_j\}_{j=1}^{n}$ is given by $v_j/s_j \geq v_i/(s_i + 1)$ for all i and j, where we define $v_j/0 = \infty$ including the case $v_j = 0$.

When $\{s_j\}$ is not unique, a version is chosen by lot in practice. The definition above may look different from a usual one, but if we consider the meaning of taking the S largest values of $\{v_j/l\}_{j=1,2,\ldots,n}^{l=1,2,\ldots}$, then we can easily understand that this is just the same. This definition is equivalent to maximize $r := \min_j (v_j/s_j)$ with respect to $\{s_j\}$. Denoting $\varepsilon_j := s_j - Sp_j$, this is equivalent to minimize[7] $\max_j (\varepsilon_j/Sp_j)$, where

[4] It seems better to assume that $v_j > 0$ because at least the candidates of the party P_j vote for their own party. Mathematically, however, it is better to allow $v_j = 0$. Otherwise, the inequalities in Theorems 1 and 2 are not generally the best.

[5] Mathematically, it is nonessential that v_j is an integer. When v_j's ($j = 1, \ldots, n$) are rational numbers, multiplying them by an adequate constant, we may regard them as integers. In Appendix B, for mathematical convenience, I take v_j's that are not integers.

[6] Strictly speaking, this definition makes sense only if there is not j such that the number of individual candidates of the party P_j is less than s_j defined above. Here I assume this. The actual data satisfy this.

[7] Lijphart and Gibberd (1977, p. 235) say that the d'Hondt system minimizes $L := \sum_{j=1}^{n} p_j/(s_j + 1)$, but this is not correct. For example, let $S = 5$, $n = 2$, $v_1 = 100$, and $v_2 = 19$. Then, in the d'Hondt system, $s_1 = 5$ and $s_2 = 0$, but L takes its minimum value at $s_1 = 4$ and $s_2 = 1$.

$\varepsilon_j/0 = 0$ ($\varepsilon_j = 0$), $= \infty$ ($\varepsilon_j \neq 0$). Here, ε_j is the absolute error (the seat bonus) of the seats of the party P_j compared with those of the perfect PR, and ε_j/Sp_j is the relative error. I think that this is a good method, but the reader might object to it. On this point, see Appendix A. Here I use a Greek letter for a variable that signifies a measure of a difference in a sense from the perfect PR. Clearly $\sum_{j=1}^{n} \varepsilon_j = 0$ holds.

By the definition of r, we have $v_j/s_j \geq r \geq v_j/(s_j+1)$, so there exists a unique θ_j (for a fixed version $\{s_j\}$) satisfying

$$v_j = r(s_j + \theta_j). \tag{1}$$

Then $0 \leq \theta_j \leq 1$ for all j, and $\theta_{j_0} = 0 \neq s_{j_0}$ for some j_0.

Conversely, assume that v_j's are expressed as $v_j = r_*(s_j + \eta_j)$ ($j = 1, 2, \ldots, n$), where s_j is a nonnegative integer, r_* is a constant, $\sum_{j=1}^{n} s_j = S$, and $0 \leq \eta_j \leq 1$. Note that j_0 satisfying $\eta_{j_0} = 0 \neq s_{j_0}$ does not necessarily exist. Then $\{s_j\}$ is a sequence of the numbers of seats (see footnote 1) because

$$\frac{v_j}{s_j} = \frac{r_*(s_j + \eta_j)}{s_j} \geq r_* \geq \frac{r_*(s_i + \eta_i)}{s_i + 1} = \frac{v_i}{s_i + 1} \quad \text{for all } i \text{ and } j.$$

We can derive $r_* \leq r$, since for j_0 satisfying $\theta_{j_0} = 0 \neq s_{j_0}$, we have $v_{j_0} = rs_{j_0} = r_*(s_{j_0} + \eta_{j_0})$. Another version of $\{s_j\}$ exists if and only if $\eta_{j_0} = 0 \neq s_{j_0}$ and $\eta_{j_1} = 1$ for some j_0 and j_1. Then, fixing j_0 and j_1, and letting $\dot{s}_j = s_j - 1$ ($j = j_0$), $= s_j + 1$ ($j = j_1$), $= s_j$ (otherwise), we get another version of seats $\{\dot{s}_j\}$. Here I use a dot to signify another version. Any other version can be expressed as this form or by repeating this process. Denoting $\Theta := \sum_{j=1}^{n} \theta_j$, we have $0 \leq \Theta \leq n - 1$. Note that r is uniquely determined and so is Θ even if $\{s_j\}$ is not uniquely determined.

There is an important meaning of r concerned with the essence of the representation system. Consider that when one votes for the party P_j, it means that one expresses one's will to have its member attend the Diet instead of one. Then a member selected in the PR system attends the Diet instead of r voters, and we can regard $r\theta_j$ as the number of wasted votes to the party P_j, and $r\Theta$ is the total of them. Those who vote for the party P_j can regard θ_j as a measure of regret for not winning another seat. In the following discussion, θ_j plays an important role. There is no influence if the party P_j loses less than $r\theta_j$ votes. Note that "the party P_j loses less than $r\theta_j$ votes" means that less than $r\theta_j$ voters for the party P_j abstain from voting, not meaning that they vote for other parties. Moreover, if there is not j_1 satisfying $\theta_{j_1} = 1$, then there is no influence even if the party P_j loses $r\theta_j$ votes. There is no influence if the party P_j increases less than $r(1-\theta_j)$ votes. If it increases exactly $r(1-\theta_j)$ votes and $\theta_j \neq 0$, then in one version there is no influence, while in another version, it increases exactly one seat. If it increases more than $r(1-\theta_j)$ votes and $\theta_{j_0} = 0 \neq s_{j_0}$ for some $j_0 \neq j$, then it increases more than one seat. For the actual data, maximum value of θ_j is 0.99501, which is of the JCP in the Kita-Kanto bloc (**K.Kanto** in Table 4). In fact, anonymous authors (1996c) in the JCP point out that if it got 1,205 more votes, it would win one more seat and defeat one candidate in the MIN. The second largest value of θ_j is 0.95998, which is of the NPS in the Kinki bloc. At present, seeing θ_j's of the *falsified* data, the reader can easily imagine how I have made the *falsified* data.

The reader might not agree to regard θ_j as a measure of a difference in a sense from the perfect PR. Then regard θ_j as a mathematical tool and do not consider a meaning of it. Still, it plays an important role.

Summing up the equality (1) with respect to j, we have

$$V = r(S + \Theta). \tag{2}$$

Therefore,

$$\frac{V}{S+n-1} \leq r = \frac{V}{S+\Theta} \leq \frac{V}{S} \tag{3}$$

holds. It is ideal that $r = V/S$ by considering the meaning of r. If $n \ll S$, then $r \approx V/S$. Otherwise, there is a possibility that $r \ll V/S$. Dividing a constituency into blocs makes S small and causes this possibility. For the actual data, r for the constituency covering the whole nation (say $r^{(\mathrm{N})}$) is larger than r in any bloc (say $r^{(k)}$ for the bloc $\mathrm{B}^{(k)}$ for $k = 1, 2, \ldots, b$), that is, $r^{(k)} < r^{(\mathrm{N})}$ for all $k = 1, 2, \ldots, b$. The value $r^{(k)}$ where $\mathrm{B}^{(k)}$ is the Shikoku bloc, is the smallest. In fact, $r^{(\mathrm{N})} \approx 272{,}865$, and in this bloc $\mathrm{B}^{(k)}$, we see $r^{(k)} = 227{,}014$. I shall similarly use $S^{(k)}$, $n^{(k)}$ etc. Then, in this bloc, $S^{(k)} = 7$ and $n^{(k)} = 6$, which are far from $n \ll S$.

Indeed, $V^{(k)}/(S^{(k)}+n^{(k)}-1) \approx 156{,}928$, so $r^{(k)}$ can take a much smaller value than the actual one. For the detailed data, see the last part of Table 4 in Appendix C. Note that $r^{(k)} < r^{(\mathrm{N})}$ does not generally hold for the *falsified* data. Next, I shall check the numbers of wasted votes for the actual data. We see that $\sum_{k=1}^{b} r^{(k)}\Theta^{(k)}$ is more than 7 times as large as $r^{(\mathrm{N})}\Theta^{(\mathrm{N})}$, so the blocs made a great number of wasted votes. For the *falsified* data, $\sum_{k=1}^{b} r^{(k)}\Theta^{(k)}$ is much smaller than the actual one, because I have artificially made $\theta_j^{(k)} \approx 0$ for all parties except the LDP.

Dividing the equality (1) by (2), we get $p_j = (s_j+\theta_j)/(S+\Theta)$. Solving this with respect to s_j, and subtracting Sp_j, we get the following formulae:

Lemma 1. *The following equalities hold:*

$$s_j = (S+\Theta)p_j - \theta_j, \quad \varepsilon_j = \Theta p_j - \theta_j.$$

Besides, $\varepsilon_j = 0$ for all j if and only if $\Theta = 0$.

They are important formulae for the following discussion. For a fixed j, if $\theta_j = 0 \neq \Theta p_j$, then $\varepsilon_j > 0$. However, we cannot conclude that there is a case that $\varepsilon_j > 0$ even if p_j is very small, because p_j and θ_j are not independent variables. In fact, if $0 < p_j < 1/(S+n-1)$, then $s_j = 0$ by the following theorem so $\varepsilon_j < 0$ holds.

Let $\underline{s_j}$ be the smallest value of s_j of all versions of $\{s_j\}$, and $\overline{s_j}$ the largest one. Note that $\overline{s_j} = \underline{s_j}$ or $\overline{s_j} = \underline{s_j}+1$ holds. For any a, define integers $[a]$ and $[a]_*$ by $[a] \le a < [a]+1$ and $[a]_* < a \le [a]_* + 1$, respectively. Here I use a for a variable that we need not consider a meaning of it.

Theorem 1. *If $0 < p_j < 1$, then the following inequalities hold:*

$$[(S+1)p_j] \le \overline{s_j} \le \min\{[(S+n-1)p_j], S\},$$
$$[(S+1)p_j]_* \le \underline{s_j} \le \min\{[(S+n-1)p_j]_*, S\}.$$

If $p_j = 0$, then $s_j = 0$. If $p_j = 1$, then $s_j = 1$. These bounds cannot be improved (see footnote 4) if we consider bounds that are functions of S, n, and p_j (j is fixed), and are independent of p_i ($i \neq j$).

I shall generalize this in Theorem 2, and we can easily derive Theorem 1 as a special case of Theorem 2. Since the lower bounds cannot be improved, we see that the minimum p_j that the party P_j could possibly win s or more seats is $s/(S+n-1)$. For $s=1$, Rokkan (1968, p. 13) essentially pointed this out, and Rae (1971, p. 193) generalizes this.[8] Since the upper bounds cannot be improved, we see that the maximum p_j that the party P_j could fail to win at least s seats is $s/(S+1)$. Historically, see Hagenbach-Bischoff *et al.* (1884, pp. 28–29), Hagenbach-Bischoff (1888, 1908), Rae *et al.* (1971), Rae (1971, p. 193), and Lijphart and Gibberd (1977), who correct errors in Rae *et al.* (1971) and Rae (1971). For $s=1$, this is numerically stated by Yamamoto *et al.* (1996) as I quoted below Table 1. By Theorem 1, if $n \ll S$, then s_j is a good approximation of Sp_j by considering the relative error, but otherwise, there is a possibility that $s_j \gg Sp_j$. To avoid this, it is better to adopt the constituency covering the whole nation.

For the actual and the *falsified* data in Appendix C, if $p_j^{(k)} \neq 0$, then $(S^{(k)}+1)p_j^{(k)}$ and $(S^{(k)}+n^{(k)}-1)p_j^{(k)}$ are non-integers. Similar statements to this are satisfied in the following discussion for the actual and the *falsified* data. Note that $p_j^{(k)} = 0$ means that the party P_j has no candidates in the bloc $\mathrm{B}^{(k)}$.[9] So

$$\left[\left(S^{(k)}+1\right)p_j^{(k)}\right] \le s_j^{(k)} \le \min\left\{\left[\left(S^{(k)}+n^{(k)}-1\right)p_j^{(k)}\right], S^{(k)}\right\}$$

[8] In Rokkan (1968), $V-1$ should read V .

[9] Strictly speaking, "n parties $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_j, \ldots, \mathrm{P}_n$ run" should read "$n^{(\mathrm{N})}$ parties $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_j, \ldots, \mathrm{P}_{n^{(\mathrm{N})}}$ run and $n^{(k)}$ of them have candidates in the bloc $\mathrm{B}^{(k)}$" here.

holds in each bloc. The upper and the lower bounds, and $s_j^{(k)}$ are written in Tables 4 and 5 in Appendix C. For example, for the actual data in the Hokkaido bloc, the upper bound for the LDP is 3 and the lower one is 2, and the actual number of seats that the LDP won is 3, which is equal to the upper bound. This is due to p_j's for the other parties. The MIN in this bloc is the contrary case.

Summing up the inequality with respect to k and denoting $s_j^{(+)} := \sum_{k=1}^{b} s_j^{(k)}$, we have

$$\sum_{k=1}^{b} \left[\left(S^{(k)}+1\right) p_j^{(k)}\right] \leq s_j^{(+)} \leq \sum_{k=1}^{b} \min\left\{\left[\left(S^{(k)}+n^{(k)}-1\right) p_j^{(k)}\right], S^{(k)}\right\}.$$

The upper and the lower bounds, and $s_j^{(+)}$ are written at the place **Total** (in boldface) in Tables 4 and 5. We see that the differences between the upper and the lower bounds are large here. For the constituency covering the whole nation, we have

$$\left[\left(S^{(\mathrm{N})}+1\right) p_j^{(\mathrm{N})}\right] \leq s_j^{(\mathrm{N})} \leq \min\left\{\left[\left(S^{(\mathrm{N})}+n^{(\mathrm{N})}-1\right) p_j^{(\mathrm{N})}\right], S^{(\mathrm{N})}\right\}.$$

The upper and the lower bounds, and $s_j^{(\mathrm{N})}$ are written at the place **Nation** in Tables 4 and 5.

For the actual data, the LDP, the NFS, the MIN, and the JCP satisfy

$$\begin{aligned}&(\text{Lower bound for Total}) < (\text{Lower bound for Nation})\\ &< (\text{Upper bound for Nation}) < (\text{Upper bound for Total}).\end{aligned}$$

So the bounds do not explain whether the blocs give an advantage to them or not. The SDP satisfies

$$\begin{aligned}&(\text{Lower bound for Total}) < (\text{Upper bound for Total})\\ &= (\text{Lower bound for Nation}) < (\text{Upper bound for Nation}).\end{aligned}$$

So the bounds explain that the blocs do not give an advantage to it, but they do not explain that the blocs give a disadvantage to it. The NSP, the NPS, and the JR satisfy

$$(\text{Upper bound for Total}) < (\text{Lower bound for Nation}).$$

So the bounds explain that the blocs give a disadvantage to them. The DRL satisfies

$$(\text{Upper bound for Total}) = (\text{Upper bound for Nation}) = 0.$$

So it wins no seat anyway. For the *falsified* data, the bounds explain that the blocs give an advantage to the JCP, the SDP, the NSP, the NPS, and the JR. I shall consider this problem theoretically in the next section.

Next, let $G \subset \{1, 2, \ldots, n\}$. Mathematically G is an arbitrary subset, but in practice it is important when the parties P_j's ($j \in G$) try to form a coalition government. Let g be the number of elements in G, and denote $v_G := \sum_{j\in G} v_j$, $p_G := \sum_{j\in G} p_j$, $s_G := \sum_{j\in G} s_j$, and $\theta_G := \sum_{j\in G} \theta_j$. Let $\underline{s_G}$ be the smallest value of s_G of all versions of $\{s_j\}$, and $\overline{s_G}$ the largest one.[10]

Theorem 2. *If $0 < p_G < 1$, then the following inequalities hold:*

$$\max\{[(S+g)p_G]+1-g, 0\} \leq \overline{s_G} \leq \min\{[(S+n-g)p_G], S\},$$
$$\max\{[(S+g)p_G]_*+1-g, 0\} \leq \underline{s_G} \leq \min\{[(S+n-g)p_G]_*, S\}.$$

If $p_G = 0$, then $s_G = 0$. If $p_G = 1$, then $s_G = 1$. These bounds cannot be improved (see footnote 4) if we consider bounds that are functions of S, n, g, and p_G, and are independent of p_j except the dependence through p_G.

For a proof, see Appendix B. The upper bounds show that the d'Hondt system can prevent a coalition government of parties that are too small.

[10] Not necessarily $\underline{s_G} = \sum_{j\in G} \underline{s_j}$ nor $\overline{s_G} = \sum_{j\in G} \overline{s_j}$. If $G = \{1, 2, \ldots, n\}$, then $\underline{s_G} = \overline{s_G} = S$, while $\sum_{j=1}^{n} \underline{s_j} < S < \sum_{j=1}^{n} \overline{s_j}$ when $\{s_j\}$ is not unique. Generally, $\sum_{j\in G} \underline{s_j} \leq \underline{s_G} \leq \overline{s_G} \leq \sum_{j\in G} \overline{s_j}$ holds.

Numerical results are given in Tables 4 and 5. First, I consider the combination of the LDP, the SDP, and the NPS, because they form a coalition now.[11] For the actual data, the number of their seats is equally 81, both in **Total** and **Nation**, but the upper and the lower bounds do not explain this. Second, to see whether the blocs give an advantage to combined large parties, I consider the combination of the LDP and the NFS, and that of the LDP, the NFS, and the MIN. The bounds, however, do not explain that the blocs gave an advantage to the combination. Third, to see whether the blocs give a disadvantage to combined medium-sized parties that are actually given a disadvantage but won seats, I consider the combination of the JCP and the SDP. The bounds, however, do not explain that the blocs give a disadvantage to the combination. Fourth, I consider the combination of the parties that could not win a seat, that is, the combination of the NSP, the NPS, the JR, and the DRL. This time the bounds show that the blocs give a disadvantage to the combination. For the *falsified* data, the bounds show that the blocs give an advantage to the last combination.

Next, regarding p_G as a constant and g as a variable, we see that the upper bounds (weakly) decrease with respect to g. Since the lower bounds for $\overline{s_G}$ and $\underline{s_G}$ can be expressed as $\max\{[(Sp_G+1)-(1-p_G)g],0\}$ and $\max\{[(Sp_G+1)-(1-p_G)g]_*,0\}$, respectively, we see that they also decrease with respect to g.

We can consider the case that the parties P_j's ($j \in G$) are merged into a party P_G. It is considered that the d'Hondt system favors mergers of parties. However, I have not found its mathematical rigorous proof in literature. Sainte-Laguë (1910) points this out, but he does not give a rigorous proof. He says, "to show this we consider the calculus of the most probable values of the numbers of seats obtained by the different parties." Rae *et al.* (1971) essentially use this fact not only for the d'Hondt system but unjustifiably also for other systems, and Rae (1971, p. 193) generalizes their result without proofs, though Lijphart and Gibberd (1977) point out their mistake. Lijphart and Gibberd (1977) accept this fact for the d'Hondt system, but a proof is not given.

Letting $G=\{1,2,\ldots,g\}$, consider that the parties P_1, P_2, ..., P_g are merged into a party P_G. In the case that the parties P_G, P_{g+1}, P_{g+2}, ..., P_n run, *assume* that the party P_G gets $v_G=\sum_{j=1}^{g} v_j$ votes and that the party P_j still gets v_j votes for $j=g+1,\ g+2,\ \ldots,\ n$. (*Remark: This is not mere convention of notation but I really assume this.*)[12] Then we have the following:

Theorem 3. *By denoting the number of seats that the party P_j wins by s'_j for $j=G,\ g+1,\ g+2,\ \ldots,\ n$, the following inequalities hold:*

$$s_G \le s'_G \le s_G+g-1, \quad s_j-g+1 \le s'_j \le s_j \ (j=g+1,\ g+2,\ \ldots,\ n), \tag{4}$$

if either s_j (s_G) or s'_j (s'_G) is uniquely determined.

Note that even if neither is uniquely determined, we can *consider* that the inequalities (4) hold. For a rigorous statement of this and a proof, see Appendix B. This shows that a merger does not decrease seats unless losing support. If the parties P_1, P_2, ..., P_g try to form a coalition government, then they are under a handicap. For the constituency covering the whole nation, if $g \ll S$, then this handicap is small.

We can also interpret Theorem 3 as follows: First there were parties P_G, P_{g+1}, P_{g+2}, ..., P_n, but the party P_G split into g parties P_1, P_2, ..., P_g. Instead of the inequalities (4), if $s'_G \le s_G$ holds, then the party P_G can win more seats by nominal splitting, unless losing support. The nominal splitting should be done by districts because then it is easy for voters to understand, and the split parties can spare money and labor in a campaign. Thanks to Theorem 3, however, the nominal splitting does not bring more seats. I think that this is a merit of the d'Hondt system.

3. A non-probabilistic approach: Part 2

In this section, I shall consider non-probabilistically the total seats compared with the case of the constituency covering the whole nation theoretically. For the notation, we should not omit an index $^{(k)}$,

[11] However, it is not a true coalition because only the LDP forms the Cabinet after the election. Before the election, it was a true coalition since the LDP, the SDP, and the NPS formed the Cabinet.

[12] This does not hold even approximately if, for example $g=2$, the supporters of the party P_1 become angry at its merger with the party P_2, and the supporters of the party P_2 become angry at its merger with the party P_1.

which signifies the bloc $\mathrm{B}^{(k)}$, an index $^{(+)}$, which signifies the total with respect to the blocs, nor an index $^{(\mathrm{N})}$, which signifies the constituency covering the whole nation. Remember that $s_j^{(+)} := \sum_{k=1}^{b} s_j^{(k)}$. Note that $S^{(\mathrm{N})} := \sum_{k=1}^{b} S^{(k)}$, $v_j^{(\mathrm{N})} := \sum_{k=1}^{b} v_j^{(k)}$, $V^{(\mathrm{N})} := \sum_{j=1}^{n^{(\mathrm{N})}} v_j^{(\mathrm{N})} = \sum_{k=1}^{b} V^{(k)}$, and $p_j^{(\mathrm{N})} := v_j^{(\mathrm{N})}/V^{(\mathrm{N})}$. It may look natural to write $S^{(+)}$ instead of $S^{(\mathrm{N})}$, but we need it to calculate seats for the constituency covering the whole nation, so I write $S^{(\mathrm{N})}$. It is similar for $v_j^{(\mathrm{N})}$, $V^{(\mathrm{N})}$, and $p_j^{(\mathrm{N})}$. Denote $\varepsilon_j^{(+)} := s_j^{(+)} - S^{(\mathrm{N})}p_j^{(\mathrm{N})}$.

Fix j and assume that $p_j^{(k)}$ is approximately independent of k, that is, $p_j^{(k)} \approx p_j$ (say). Then $v_j^{(k)} \approx V^{(k)}p_j$ holds, and summing this up with respect to k, we have $v_j^{(\mathrm{N})} \approx V^{(\mathrm{N})}p_j$, so $p_j^{(\mathrm{N})} \approx p_j$. Therefore, we get

$$\sum_{k=1}^{b} S^{(k)}p_j^{(k)} \approx \left(\sum_{k=1}^{b} S^{(k)}\right) p_j = S^{(\mathrm{N})}p_j \approx S^{(\mathrm{N})}p_j^{(\mathrm{N})}.$$

Numerically, $S^{(\mathrm{N})}p_j^{(\mathrm{N})}$ is given at the place "Perfect" of **Total** and **Nation**, and $\sum_{k=1}^{b} S^{(k)}p_j^{(k)}$ is given under the place **Total**.

Instead of the assumption above, assume that $V^{(k)}/S^{(k)}$ is approximately independent of k, that is, $V^{(k)}/S^{(k)} \approx c$ (say). This holds if malapportionment does not arise and the absolute proportions of the valid ballots are approximately independent of the blocs. Then $V^{(k)} \approx cS^{(k)}$ holds, and summing this up with respect to k, we have $V^{(\mathrm{N})} \approx cS^{(\mathrm{N})}$, so $V^{(\mathrm{N})}/S^{(\mathrm{N})} \approx c$. Therefore, we get

$$\sum_{k=1}^{b} S^{(k)}p_j^{(k)} = \sum_{k=1}^{b} \frac{S^{(k)}v_j^{(k)}}{V^{(k)}} \approx \frac{\sum_{k=1}^{b} v_j^{(k)}}{c} = \frac{v_j^{(\mathrm{N})}}{c} \approx \frac{S^{(\mathrm{N})}v_j^{(\mathrm{N})}}{V^{(\mathrm{N})}} = S^{(\mathrm{N})}p_j^{(\mathrm{N})}.$$

Numerically, $V^{(k)}/S^{(k)}$ is given at the last part of Tables 4 and 5. Note that the actual data are of the first election under the new system, so it is natural that malapportionment does not arise.

Hence altogether, if either $p_j^{(k)}$ or $V^{(k)}/S^{(k)}$ is approximately independent of k, then we have $\sum_{k=1}^{b} S^{(k)}p_j^{(k)} \approx S^{(\mathrm{N})}p_j^{(\mathrm{N})}$, so we get

$$\varepsilon_j^{(+)} = s_j^{(+)} - S^{(\mathrm{N})}p_j^{(\mathrm{N})} \approx \sum_{k=1}^{b} s_j^{(k)} - \sum_{k=1}^{b} S^{(k)}p_j^{(k)} = \sum_{k=1}^{b} \left(s_j^{(k)} - S^{(k)}p_j^{(k)}\right) = \sum_{k=1}^{b} \varepsilon_j^{(k)}.$$

This approximation is, however, important for a probabilistic approach. For the question whether the d'Hont system with blocs gives an advantage or not, a non-probabilistic approach is useful when it is a question whether the party wins a seat, or when a party is supported in only one bloc. As I noted below Theorem 1, to win a seat, $p_j^{(\mathrm{N})} \geq 1/(S^{(\mathrm{N})} + n^{(\mathrm{N})} - 1)$ is necessary and $p_j^{(\mathrm{N})} > 1/(S^{(\mathrm{N})} + 1)$ is sufficient. We can easily see it by Theorem 1. Under blocs, $p_j^{(k)} \geq 1/(S^{(k)} + n^{(k)} - 1)$ for some k is necessary and $p_j^{(k)} > 1/(S^{(k)} + 1)$ for some k is sufficient. If $p_j^{(k)}$ is approximately independent of k, then the blocs do not give an advantage as long as it is a question whether the party wins a seat or not. The explanation by Yamamoto *et al.* (1996) quoted below Table 1 makes sense then. Otherwise, blocs may give an advantage. To see this, we may let $j = 1$. Consider that the party P_1 is supported in only one bloc (say $\mathrm{B}^{(1)}$) and it gets no votes in other blocs. For the constituency covering the whole nation, to win a seat, $v_1^{(1)} \geq V^{(\mathrm{N})}/(S^{(\mathrm{N})} + n^{(\mathrm{N})} - 1)$ is necessary and $v_1^{(1)} > V^{(\mathrm{N})}/(S^{(\mathrm{N})} + 1)$ is sufficient. Under blocs, $v_1^{(1)} \geq V^{(1)}/(S^{(1)} + n^{(1)} - 1)$ is necessary and $v_1^{(1)} > V^{(1)}/(S^{(1)} + 1)$ is sufficient. In addition, assume that $V^{(k)}/S^{(k)} \approx c$, $n^{(\mathrm{N})} \ll S^{(\mathrm{N})}$ and $S^{(1)}$ is not so large. Then, to win a seat in the constituency covering the whole nation is approximately equivalent to $v_1^{(1)} > c$, while the condition to win a seat under the blocs is much weaker than $v_1^{(1)} > c$. For further details, under the assumptions above,

$$c \approx \frac{V^{(\mathrm{N})}}{S^{(\mathrm{N})} + n^{(\mathrm{N})} - 1} > \frac{V^{(1)}}{S^{(1)} + 1}, \quad \text{i.e.,} \quad \frac{1}{c} \approx \frac{S^{(\mathrm{N})} + n^{(\mathrm{N})} - 1}{V^{(\mathrm{N})}} < \frac{S^{(1)} + 1}{V^{(1)}}$$

holds. Numerically, see the last part of Tables 4 and 5. For the actual data, 8 blocs satisfy this inequality, while 3 blocs do not. Under this inequality, we have

$$s_1^{(\mathrm{N})} \leq \left[\frac{S^{(\mathrm{N})}+n^{(\mathrm{N})}-1}{V^{(\mathrm{N})}} v_1^{(\mathrm{N})}\right] = \left[\frac{S^{(\mathrm{N})}+n^{(\mathrm{N})}-1}{V^{(\mathrm{N})}} v_1^{(1)}\right] \leq \left[\frac{S^{(1)}+1}{V^{(1)}} v_1^{(1)}\right]_* \leq s_1^{(1)} = s_1^{(+)},$$

so $s_1^{(\mathrm{N})} \leq s_1^{(+)}$. Hence the blocs do not give a disadvantage for a party supported in only one bloc. I have made the *falsified* data of the NPS, the NPS, and the JR considering this. Therefore, for a very small party to win a seat in the d'Hont system with blocs, it is better to be supported in its own territory. So this system is a hotbed of bribery.

4. A probabilistic approach to seats in a fixed constituency

In this section, I shall consider probabilistically in a fixed constituency that we select S seats. I regard v_j as the realization of a random variable $\tilde{v}_j$. Here I use a tilde to signify a random variable.[13] Note that the following discussion is not mere application of a usual statistical method. The reader might object to a probabilistic approach. In fact, this problem is concerned with a philosophical problem of mathematical statistics. Extreme non-Bayesians object to it because an election is not carried out under a random sampling. They do not consider the probability of, for example, the event that the DRL gets (or will get) more votes than the LDP. They do not say that the probability that $\{s_j\}$ is not uniquely determined is very small. On the other hand, extreme Bayesians, before an election, consider as follows:

"I do not know what others vote for. So the number of votes that each party gets is a random variable, and its distribution is determined by my subjectivity. It does not matter even if the distribution for another person is different from mine. Of course I can consider the probability of the event that the DRL will get more votes than the LDP. For me, for example, it is 0.03. For one who has no knowledge of Japanese politics, it is 0.5. After I see the election returns, v_j will be a constant for me because I shall know it."

Another standpoint is as follows: Regarding human beings as products made by a machine, we can consider that each elector independently votes for the party P_j with probability u_j^* $(j = 1, 2, \ldots, n)$, and abstains from voting or makes invalid voting with probability u_0^*, where u_j^* is an unknown constant satisfying $u_j^* \geq 0$ $(j = 0, 1, 2, \ldots, n)$ and $\sum_{j=0}^{n} u_j^* = 1$.

I adopt neither standpoint in the following discussion. Consider the following imaginary experiments. We carry out an election. After carrying it out, we carry an election again. Assume that, between the two elections, no information is added. Then, one who votes with belief, votes for the same party in the two elections. One who votes without belief, might vote for different parties. Consider continuing elections repeatedly without added information, and regard the actual election as one of the elections in the imaginary experiments, then non-Bayesians can regard v_j as the realization of a random variable $\tilde{v}_j$. I shall similarly use $\tilde{p}_j$, $\tilde{s}_j$, $\tilde{\varepsilon}_j$, $\tilde{\theta}_j$, and $\tilde{\Theta}$. By the equality of ε_j in Lemma 1, we have $E(\tilde{\varepsilon}_j) = E(\tilde{\Theta}\tilde{p}_j) - E(\tilde{\theta}_j)$. I assume the following:

Assumption 1. The random variable $\tilde{p}_j$ can change only a little, that is, $\tilde{p}_j \approx p_j^* := E(\tilde{p}_j)$, but not too little.

Note that p_j^* is an unknown constant, not a random variable.[14] I use a superscript asterisk for a constant that we cannot observe. In mathematical statistics, this is called a parameter (or a function of parameters) and usually denoted by a Greek letter. Then we have $E(\tilde{\varepsilon}_j) \approx E(\tilde{\Theta})p_j^* - E(\tilde{\theta}_j)$. I denote $\theta_j^* := E(\tilde{\theta}_j)$, and similarly use Θ^*, ε_j^*, and s_j^*. In this notation, we get $\varepsilon_j^* \approx \Theta^* p_j^* - \theta_j^*$. The reader might consider that θ_j^* is independent of j, or approximately so. However, this is inadequate. I further assume the followings:

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[13] Conventionally, we use a capital letter, but I avoid this here because I use a capital letter for a variable expressed as a total with respect to $j = 1, 2, \ldots, n$.

[14] This time, Bayesians object. They consider that an unknown thing is a random variable. Here, there is no problem even if p_j^* is known, but there is a problem when we apply the following results to the actual data. In their standpoint, one determines p_j^* by one's subjectivity, not one estimates it by the data.

Assumption 2. $S + n \ll \tilde{V} \not\approx \overline{V}$, where $\overline{V}$ is the number of members of the electorate.
Assumption 3. The magnitude S is not too small.
Assumption 4. We can approximately consider that $\{s_j\}$ is uniquely determined and that so is j_0 satisfying $\theta_{j_0} = 0$. That is, the probability of the exceptional event is very small. Define a random variable $\tilde{\jmath}_0$ by $\tilde{\theta}_{\tilde{\jmath}_0} = 0$.
Assumption 5. As a mathematical tool, consider that S is also the realization of a random variable $\tilde{S}$. Assume that $P[\tilde{S} = S] = 1/(\overline{S} - \underline{S} + 1)$ for $S = \underline{S}, \underline{S}+1, \ldots, \overline{S}$, where $\underline{S} \ll \overline{S}$, though $\underline{S}$ is not so small, and $\overline{S}$ is not too large. Then the random variables $\tilde{\jmath}_0$ and $\tilde{S}$ are approximately independent. That is, we can approximately use Fisher's fiducial argument to get $P[\tilde{\theta}_j = 0]$ by regarding S as a random variable.
Assumption 6. Fix v_j that $\tilde{v}_j$ can take. (Then V and r are determined correspondingly.) For any fixed $i = 1, 2, \ldots, n$, the conditional distribution of $\tilde{v}_i$ under the conditions $v_i - a < \tilde{v}_i < v_i + r - a$ $(0 \le a \le r)$ and $\tilde{v}_j = v_j$ $(j \neq i)$ is approximately[15] the uniform distribution on the interval $(v_i - a, v_i + r - a)$ if v_i is not too small.

Then we have the following lemma:

Lemma 2. *Under Assumptions 1 to 6, by letting*

$$M^* := \{j : p_j^* \ge t^*\} \text{ for some small } t^* > 0, \quad p_j^{**} := \frac{p_j^*}{p_{M^*}^*} \quad \text{for } j \in M^*,$$

the following approximation is satisfied:

$$\theta_j^* \approx \frac{1 - p_j^{**}}{2} \quad \text{for } j \in M^*.$$

For a proof, see Appendix B. There is a problem how to determine t^*. Roughly speaking, $\tilde{p}_j < t^*$ means that the party P_j can win no seat anyway. Let m^* be the number of elements in M^*. Denote $w^* := \sum_{j \notin M^*} \theta_j^*$ and $m^{**} := m^* + 2w^*$. Then we have

$$\Theta^* = \sum_{j=1}^{n} \theta_j^* \approx \sum_{j \in M^*} \frac{1 - p_j^{**}}{2} + \sum_{j \notin M^*} \theta_j^* = \frac{m^* - 1}{2} + w^* = \frac{m^{**} - 1}{2},$$

therefore, we have the following theorem:

Theorem 4. *Under Assumptions 1 to 6 and the notation above, the following approximation holds:*

$$\varepsilon_j^* \approx \frac{m^{**} - 1}{2} p_j^* - \frac{1 - p_j^{**}}{2} \quad \text{for } j \in M^*.$$

In particular, if $w^* \approx 0$, then

$$\varepsilon_j^* \approx \frac{m^* p_j^* - 1}{2} \quad \text{for } j \in M^*.$$

These formulae show that the d'Hont system gives an advantage to large parties *in the sense of the expectation.* However, the right-hand sides in the two formulae above depend on S only through t^*.[16] This

[15] As often happens when we use a continuous distribution as an approximation, this is never exactly the uniform distribution because $\tilde{v}_i$ can take only integers. However, the length of the interval is r, which satisfies the inequality (3). By Assumption 2 ($S + n \ll \tilde{V}$), we see that r is sufficiently large. So it is natural to use a continuous distribution as an approximation.

[16] Strictly speaking, this is under the assumption that candidates and voting are independent of S. This is not satisfied if one considers, for example, "To tell the truth, I support the party P_1, but I think that it can win no seat anyway because S is too small. So I vote for a larger party."

has an important meaning. First, if S is large, the tendency to give an advantage to large parties is small by considering the relative error. Second, remember that t^* is a measure of excluding small parties. Regarding it as a function of S, it decreases with respect to S. So a large party can make "magnitude gerrymander" by letting $S = 10$ instead of $S = 100$, but it cannot make magnitude gerrymander by letting $S = 101$ instead of $S = 100$.

The standpoint based on u_j^* seems to justify the argument above, but this is not correct. A reason is not philosophical but mathematical (see Appendix B). I shall apply the results above to the actual data. Remember that the values expressed with an asterisk are unknown. So I have to estimate them. I have assumed that $\tilde{p}_j \approx p_j^*$, and I regard the actual data p_j as the realization of $\tilde{p}_j$. So I shall use p_j as an estimate[17] of p_j^*. I write $\widehat{p_j^*} := p_j$ to express this, where $\widehat{p_j^*}$ signifies an estimate of p_j^*. Next, let $\widehat{M^*} := \{j : s_j + \theta_j \geq 1/2\}$. I do so for convenience' sake, but for a reason, see Appendix B. I estimate m^* and p_j^{**} accordingly,[18] that is, let $\widehat{m^*}$ be the number of elements in $\widehat{M^*}$, and $\widehat{p_j^{**}} := \widehat{p_j^*}/\sum_{i\in\widehat{M^*}} \widehat{p_i^*}$. Next I shall estimate θ_j^*. For $j \in \widehat{M^*}$, according to Lemma 2, I let $\widehat{\theta_j^*} := (1-\widehat{p_j^{**}})/2$. For $j \notin \widehat{M^*}$, the non-probabilistic approach is useful so estimating θ_j^* is not important in itself, but the formulae in Theorem 4 depend on m^*, which depends on θ_i^*'s ($i \notin M^*$) for any fixed j. So we have to estimate θ_j^* even if $j \notin \widehat{M^*}$. Though this is also for convenience' sake, I let $\widehat{\theta_j^*} := \theta_j$ then. Hence altogether,

$$\widehat{\theta_j^*} := \begin{cases} \dfrac{1-\widehat{p_j^{**}}}{2} & \text{for } j \in \widehat{M^*}, \\ \theta_j & \text{for } j \notin \widehat{M^*}. \end{cases}$$

I estimate w^* and m^{**}, accordingly, that is, $\widehat{w^*} := \sum_{j\notin\widehat{M^*}} \widehat{p_j^*}$ and $\widehat{m^{**}} := \widehat{m^*} + 2\widehat{w^*}$. Next, I estimate ε_j^* according to the first formula in Theorem 4, that is,

$$\widehat{\varepsilon_j^*} := \frac{\widehat{m^{**}}-1}{2}\widehat{p_j^*} - \frac{1-\widehat{p_j^{**}}}{2} \quad \text{for } j \in \widehat{M^*}.$$

The assumption for the second formula in Theorem 4 is so strong that I do not use it. For $j \notin \widehat{M^*}$, it is not important to estimate ε_j^* because the non-probabilistic approach is useful. To get estimates $\widehat{\varepsilon_j^*}$'s ($j = 1, 2, \ldots, n$) satisfying $\sum_{j=1}^n \widehat{\varepsilon_j^*} = 0$, however, we should let

$$\widehat{\varepsilon_j^*} := \frac{\widehat{m^{**}}-1}{2}\widehat{p_j^*} - \widehat{\theta_j^*} \quad \text{for } j \notin \widehat{M^*}.$$

Next, according to $s_j^* = Sp_j^* + \varepsilon_j^*$, I estimate s_j^*, that is, $\widehat{s_j^*} := S\widehat{p_j^*} + \widehat{\varepsilon_j^*}$.

[17] In convention of mathematical statistics, the random variable $\tilde{p}_j$ is called an *estimator* of p_j^*, and the realization p_j of the estimator $\tilde{p}_j$ is called an *estimate*.

[18] The argument where I say "according(ly)" is not generally justified in mathematical statistics. Estimating $h(a^*)$ is different from estimating a^*, where a^* is a parameter (conventionally θ, but different from θ_j here) in a general case. In fact, if $\widehat{a^*}$ is an unbiased estimator of a^*, then $(\widehat{a^*})^2$ is *not* an unbiased estimator of $(a^*)^2$ except trivial cases. Here, however, because $\tilde{p}_j \approx p_j^*$ is assumed, $h(\tilde{p}_1, \ldots, \tilde{p}_n) \approx h(p_1^*, \ldots, p_n^*)$ follows for a continuous function h whose value does not move violently. The problem is $\widehat{M^*}$ and $\widehat{\theta_j^*}$ ($j \notin \widehat{M^*}$), but they do not affect so much unless there are many parties near or under the borderline.

In Tables 4, the values of $\widehat{s_j^*}$, which are called "Estimates", are given in all blocs.[19] For example, in the Kyushu bloc for the MIN, s_j(Seats) $= 3$ while $\widehat{s_j^*}$(Estimate) ≈ 2.39. We can consider that this is good luck for the MIN there. In the K.Kanto (Kita-Kanto) bloc for the JCP, s_j(Seats) $= 2$ while $\widehat{s_j^*}$(Estimate) ≈ 2.62. We can consider that this is bad luck for the JCP there. In almost (but not all) cases in a bloc, s_j is the integer given by rounding off $\widehat{s_j^*}$. I have "unjustifiably" calculated $\widehat{s_j^*}$ for the *falsified* data. We see that $s_j < \widehat{s_j^*}$ for the LDP in all blocs, but it is natural because I have artificially given a disadvantage to the largest party. For the actual data, $\widehat{s_j^*}$ can be negative though its absolute value is small. This is due to the convenience' sake to define $\widehat{s_j^*}$ for $j \notin \widehat{M^*}$. Such a small contradiction naturally arises when we consider approximations. In the Tokyo bloc, for the SDP, both the upper and the lower bounds equal 1, but $\widehat{s_j^*} \approx 0.68 \not\approx 1$. This seems a contradiction, but it is not so. To get the bounds, we regard p_j for the SDP as a constant, that is, we do not consider that the SDP could get higher or lower proportion of votes. In contrast, to obtain $\widehat{s_j^*}$, we regard p_j as the realization of a random variable, that is, we consider that the SDP could get higher or lower proportion of votes. For a set G, I defined $\widehat{s_G^*} := \sum_{j\in G} \widehat{s_j^*}$ in any bloc.

5. A probabilistic approach to the total seats compared with the case of the constituency covering the whole nation

In this section, I shall consider probabilistically the total seats compared with the case of the constituency covering the whole nation.

We have seen that $\varepsilon_j^{(+)} \approx \sum_{k=1}^{b} \varepsilon_j^{(k)}$ holds if either $p_j^{(k)}$ or $V^{(k)}/S^{(k)}$ is approximately independent of k. I have also announced that this approximation is important for a probabilistic approach. I define $\varepsilon_j^{(+)*}$, $m^{(k)**}$ etc. corresponding to the non-probabilistic approach. For example, corresponding to define $\varepsilon_j^{(+)} := s_j^{(+)} - S^{(\mathrm{N})}p_j^{(\mathrm{N})}$, I define $\varepsilon_j^{(+)*} := s_j^{(+)*} - S^{(\mathrm{N})}p_j^{(\mathrm{N})*}$. Assume that either $p_j^{(k)}$ or $V^{(k)}/S^{(k)}$ is approximately independent of k. Then $\varepsilon_j^{(+)} \approx \sum_{k=1}^{b} \varepsilon_j^{(k)}$ holds, so we get

$$\varepsilon_j^{(+)*} \approx \sum_{k=1}^{b} \left(\frac{m^{(k)**}-1}{2} p_j^{(k)*} - \frac{1-p_j^{(k)**}}{2} \right) \quad \text{for } j \in \bigcap_{k=1}^{b} M^{(k)*}.$$

In particular, if $w^{(k)*} \approx 0$, then

$$\varepsilon_j^{(+)*} \approx \sum_{k=1}^{b} \frac{m^{(k)*}p_j^{(k)*}-1}{2} \quad \text{for } j \in \bigcap_{k=1}^{b} M^{(k)*}.$$

Moreover, if $M^{(k)*}$ is independent of k, and $p_j^{(k)*}$ and $m^{(k)*}$ are approximately independent of k, that is, $M^{(k)*} = M^*$, $p_j^{(k)*} \approx p_j^*$, and $m^{(k)*} \approx m^*$ (say), then

$$\varepsilon_j^{(+)*} \approx b\frac{m^* p_j^* - 1}{2} \quad \text{for } j \in M^*.$$

This formula has an important meaning, though the assumptions are made in order to simplify the discussion and are too strong to apply to the actual data. We have already seen that the d'Hondt system gives an advantage to large parties in the sense of the expectation. And this formula shows that dividing a constituency into blocs exaggerates this. Even for the constituency covering the whole nation, the d'Hondt system gives an advantage to large parties in the sense above, but I think that this is unavoidable. If we

[19] Strictly speaking, I must admit that it is not reasonable enough to apply the probabilistic approach in a bloc where $S^{(k)}$ is not so large, especially in the Shikoku bloc ($S^{(k)} = 7$).

try to avoid this, we should give a seat for even a very small party. However, I object to exaggerate this by dividing a constituency into blocs.

I shall consider numerically. Here, I do not use $\varepsilon_j^{(+)} \approx \sum_{k=1}^{b} \varepsilon_j^{(k)}$. I rewrite $\widehat{s_j^*}$ in the bloc $\mathrm{B}^{(k)}$by $\widehat{s_j^{(k)*}}$. I estimate $s_j^{(+)*}$ by $\widehat{s_j^{(+)*}} := \sum_{k=1}^{b} \widehat{s_j^{(k)*}}$, which is "Estimate" in **Total**. I write $s_j^{(\mathrm{N})*}$ for the constituency covering the whole nation, which is "Estimate" in **Nation**.

I shall consider the LDP. We see $Sp_j^{(\mathrm{N})} = 65.53$ ("Perfect", written in both **Total** and **Nation**). For the constituency covering the whole nation, $\widehat{s_j^{(\mathrm{N})*}}$(Estimate) ≈ 66.36, so it is advantageous a little in the sense of the expectation, and since $s_j^{(\mathrm{N})}$(Seats) $= 66$, it is somewhat bad luck, but a little advantageous. For the total number of seats, $s_j^{(+)}$(Seats) $= 70$ and $Sp_j^{(\mathrm{N})} \approx 65.53$, so the LDP is very advantageous, but the upper and the lower bounds do not explain this. However, $\widehat{s_j^{(\mathrm{N})*}} \approx 71.63$, so this system could give more advantage to the LDP, but it was bad luck for the LDP that this system gave a smaller advantage.

We can explain for other parties, too. So we can see the d'Hondt system gives an advantage to large parties in the sense of the expectation a little, and that the blocs exaggerate this.

Appendix A

Here, I shall rejoin the following to some presumable objections to the d'Hont system.

Objection 1: In the d'Hont system, minimizing $\max_j (\varepsilon_j / Sp_j)$, we can prevent $s_j \gg Sp_j$, but it is irrational not to prevent $s_j \ll Sp_j$.

Rejoinder 1: Because $\sum_{j=1}^{n} s_j = S$, where S is a given constant, preventing $s_j \gg Sp_j$, we can also prevent $s_j \ll Sp_j$. In fact, Theorem 1 holds.

Objection 2: Even so, it is better to minimize $\max_j (|\varepsilon_j| / Sp_j)$.

Rejoinder 2: In this method, however, it becomes oversensitive to seats of small parties because $|\varepsilon_j|/Sp_j = 1$ if $s_j = 0 \neq p_j$. For example, let $S = n = v_1 \geq 3$ and $v_2 = v_3 = \cdots = v_n = 1$. Then $V = 2n - 1$, $p_1 = n/(2n-1) > 1/2$, $p_2 = p_3 = \cdots = p_n = 1/(2n-1)$, $Sp_1 > n/2$, and $Sp_2 = Sp_3 = \cdots = Sp_n = n/(2n-1) > 1/2$. We have $|\varepsilon_j|/Sp_j < 1$ if $s_1 = s_2 = \cdots = s_n = 1$, and $|\varepsilon_j|/Sp_j \geq 1$ otherwise. Therefore, to minimize $\max_j (|\varepsilon_j|/Sp_j)$, each parties have 1 seat even more than half the votes are to the party P_1. In particular, if all the members in the Lower House were selected in this way in the constituency covering the whole nation, it is possible for the parties P_2, P_3, ..., P_n to form a coalition government.

Objection 3: Since dividing by Sp_j gives an advantage to large parties, we should not do so but minimize $\max_j |\varepsilon_j|$.

Rejoinder 3: Such a system also exists. This is essentially[20] the simple (Hare) quota and largest remainders, which is known by the paradox of Alabama. See, e.g., Nisihira (1981, p. 86, and 1990, pp. 51–60). In this method, it is easy to calculate, though it is not essential today because there are computers. In addition, consider the case that the parties P_1, ..., P_g try to form a coalition government. Then $\varepsilon_1, \ldots, \varepsilon_g$ are not important but $\max\{|\sum_{j=1}^{g} \varepsilon_j|, |\varepsilon_{g+1}|, \ldots, |\varepsilon_n|\}$ is important. If we minimize $\max_j |\varepsilon_j|$, however, then $|\sum_{j=1}^{g} \varepsilon_j|$ is not always small.

Objection 4: The fact that the upper bounds in the inequalities in Theorem 1 cannot be improved shows that the d'Hondt system is bad.

Rejoinder 4: Since the d'Hondt system minimizes $\max_j (\varepsilon_j / Sp_j)$, it minimizes $\max_j \alpha_j$, where $s_j = (S + \alpha_j)p_j$ $(p_j \neq 0)$, $\alpha_j = \infty$ $(p_j = 0 \neq s_j)$, and $\alpha_j = 0$ $(p_j = s_j = 0)$. Since, for any different system, if we consider an upper bound of the form $\overline{s_j} \leq (S + \beta)p_j$ for all choices of $\{p_j\}_{j=1}^{n}$ such that $p_j > 0$ and $\sum_{j=1}^{n} p_j = 1$, the constant β does not become smaller than $n - 1$. Surely the d'Hondt system can be a bad one unless $n \ll S$, so it is better to adopt the constituency covering the whole nation as I noted below Theorem 1.

[20] Strictly speaking, there is a problem of managing fractions on the way of calculations.

Objection 5 (Mizuki, 1967, pp. 326–327): Consider the case that $S = 11$, $n = 3$, $v_1 = 1{,}900$, $v_2 = 4{,}800$, and $v_3 = 6{,}000$. Then $s_1 = 1$, $s_2 = 4$, and $s_3 = 6$. Though the party P_3 wins 6 seats by getting 6,000 votes, the parties P_1 and P_2 together win only 5 seats by getting 6,700 votes.

Rejoinder 5: I think that it is rather a merit that the parties P_1 and P_2 together win only one less seats than the party P_3 wins. If the parties P_1 and P_2 are merged into a party $\mathrm{P}_{\{1,2\}}$, then it wins 6 seats while the party P_3 wins 5 seats. If they do not merge, the parties P_1 and P_2 together win only one less seats than the merged case. By the Theorem 3, if S is any given, then the number of seats the parties P_1 and P_2 together win is one less than, or equal to, the merged case. The true problem of this example is that S is too small. In fact, for any given S, we have $\underline{s_{\{1,2\}}} \geq [(S+2)p_{\{1,2\}}]_* - 1 = [Sp_{\{1,2\}} + (2p_{\{1,2\}}1)]_* \geq [Sp_{\{1,2\}}]_*$ by the latter inequality in Theorem 2. So if S is not small, then $s_{\{1,2\}} > s_3$, and this problem does not arise.

Objection 6: Then we should adopt a method that such a problem does not arise even if S is small.

Rejoinder 6: To avoid this, the problem is Theorem 3. We should avoid $s'_G = s_G + 1$ even if $g = 2$. Consider that a party P_G splits into two parties, each of them splits into two parties, each of them splits into two parties, and so on. Then the party P_G has become parties $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_g$, and each p_j $(j = 1, 2, \ldots, g)$ is very small. To avoid $s'_G = s_G + 1$ even if $g = 2$, then we should give a seat even if p_j $(j = 1, 2, \ldots, g)$ is very small. This is rather irrational.

Appendix B

Proof of Theorem 2. Assume that $0 < p_G < 1$. We may let $G = \{1, 2, \ldots, g\}$ $(1 \leq g \leq n-1)$. Fix a version of $\{s_j\}$. Summing up the equality of s_j in (2.1) with respect to $j \in G$, we have

$$s_G = (S + \Theta)p_G - \theta_G = (S + \theta_G + \theta_H)p_G - \theta_G = (S + \theta_H)p_G - \theta_G(1 - p_G),$$

where $H := \{g+1,\, g+2,\, \ldots,\, n\}$. Using $0 \leq \theta \leq 1$, we get

$$s_G \geq Sp_G - g(1 - p_G) = (S + g)p_G - g. \tag{5}$$

Since s_G is an integer, $s_G \geq [(S+g)p_G]_* + 1 - g$ follows. This holds for all versions of $\{s_j\}$, so we have $\underline{s_G} \geq [(S+g)p_G]_* + 1 - g$. Since $0 < p_G < 1$, the sign of equality holds in the inequality (5) if and only if $\theta_1 = \theta_2 = \cdots = \theta_g = 1$ $(g \geq 1)$ and $\theta_{g+1} = \theta_{g+2} = \cdots = \theta_n = 0$ so $\overline{s_G} > s_G$ in this case. Therefore, $\overline{s_G} > (S+g)p_G - g$ generally holds and $\overline{s_G} \geq [(S+g)p_G] + 1 - g$ follows. Clearly $\overline{s_G} \geq \underline{s_G} \geq 0$ holds. Hence we have obtained the lower bounds. To get the upper bounds, applying the lower bounds to s_H,[21] we have

$$\begin{aligned}
\overline{s_G} &= S - \underline{s_H} \\
&\leq S - \max\{[(S+n-g)p_H]_* + 1 - (n-g), 0\} \\
&= \min\{S - [(S+n-g)p_H]_* - 1 + n - g, S\} \\
&= \min\{[S - (S+n-g)p_H + n - g], S\} \\
&= \min\{[S - (S+n-g)(1-p_G) + n - g], S\} \\
&= \min\{[(S+n-g)p_G], S\},
\end{aligned}$$

and we can similarly derive $\underline{s_G} \leq \min\{[(S+n-g)p_G]_*, S\}$.

Next, we shall show that the bounds cannot be improved. We may assume that $G = \{1, 2, \ldots, g\}$ $(1 \leq g \leq n-1)$. Let $0 < p < 1$. If we show that the lower bounds cannot be improved, then we see by the proof of the upper bounds that they cannot be improved, either.

(i) Assume that $(S+g)p+1-g$ is a nonnegative integer. Let $v_1 = (S+g)p+1-g$, $v_2 = v_3 = \cdots = v_g = 1$, $v_{g+1} = (S+g)(1-p)$, and $v_{g+2} = v_{g+3} = \cdots = v_n = 0$, then they are nonnegative integers, $v_G = (S+g)p$, $V = S + g$, and $p_G = p$. Let $r_* = 1$, $s_1 = (S+g)p - g$, $s_2 = s_3 = \cdots = s_g = 0$, $s_{g+1} = (S+g)(1-p)$, $s_{g+2} = s_{g+3} = \cdots = s_n = 0$, $\eta_1 = \eta_2 = \cdots = \eta_g = 1$, and $\eta_{g+1} = \eta_{g+2} = \cdots = \eta_n = 0$. Then $\sum_{j=1}^{n} s_j = S$, $0 \leq \eta_j \leq 1$ for all j, and $s_{j_0} \neq 0$ and $\eta_{j_0} = 0$ for some j_0 hold. Hence $\{s_j\}$ is a version of seats and

[21] It is easier to derive the upper bounds directly. However, I use this method because it is useful when we show that the bounds cannot be improved.

$s_G = \max\{[(S+g)p_G]_* + 1 - g, 0\}$ is satisfied. So $\underline{s_G} = \max\{[(S+g)p_G]_* + 1 - g, 0\}$ is satisfied. (Note that $\overline{s_G} = \max\{[(S+g)p_G] + 1 - g, 0\} + (g-1)$.) Therefore, $\underline{s_G} = \max\{[(S+g)p_G]_* + 1 - g, 0\}$ is the best bound for $\underline{s_G}$ if $(S+g)p_G + 1 - g$ is a nonnegative integer.

(ii) Assume that $(S+g)p + 1 - g$ is nonnegative. Then we can take a positive number a satisfying $(S+g)p + (g-1)a < [(S+g)p] + 1$. Let $v_1 = (S+g)p + (g-1)a + 1 - g$, $v_2 = v_3 = \cdots = v_g = 1 - a$, $v_{g+1} = (S+g)(1-p)$, and $v_{g+2} = v_{g+3} = \cdots = v_n = 0$, then they are nonnegative (see footnote 5), $v_G = (S+g)p$, $V = S+g$, and $p_G = p$. Let $r_* = 1$, $s_1 = [(S+g)p + (g-1)a] + 1 - g$, $s_2 = s_3 = \cdots = s_g = 0$, $s_{g+1} = [(S+g)(1-p)]$, $s_{g+2} = s_{g+3} = \cdots = s_n = 0$, $\eta_1 = \{(S+g)p + (g-1)a\} - [(S+g)p + (g-1)a]$, $\eta_2 = \eta_3 = \cdots = \eta_g = 1 - a$, and $\eta_{g+1} = (S+g)(1-p) - [(S+g)(1-p)]$, $\eta_{g+2} = \cdots = \eta_n = 0$, then s_j and η_j satisfy the similar conditions to the case (i). Hence $\{s_j\}$ is a version of seats and $s_G = \max\{[(S+g)p_G] + 1 - g, 0\}$ is satisfied. Because $\eta_j < 1$ here, $\{s_j\}$ is uniquely determined. So $\underline{s_G} = \overline{s_G} = \max\{[(S+g)p_G] + 1 - g, 0\}$ is satisfied. Therefore, this is the best bound for $\overline{s_G}$. If $(S+g)p_G + 1 - g$ is not an integer, then $\max\{[(S+g)p_G] + 1 - g, 0\} = \max\{[(S+g)p_G]_* + 1 - g, 0\}$, so this is also the best bound for $\underline{s_G}$.

(iii) Assume that $(S+g)p + 1 - g < 0$. Let $v_1 = v_2 = \cdots = v_g = (S+g)p/g$, $v_{g+1} = (S+g)(1-p)$, and $v_{g+2} = v_{g+3} = \cdots = v_n = 0$, then they are nonnegative, $v_G = (S+g)p$, $V = S+g$, and $p_G = p$. From the assumption, $(S+g)p < g - 1$ so $v_j = (S+g)p/g < (g-1)/g < 1$ $(j = 1, 2, \ldots, g)$. On the other hand, $v_{g+1} = (S+g)(1-p) = (S+g) - (S+g)p > (S+g) - (g-1) = S + 1$, so $v_{g+1}/S > 1$. Hence $\{s_j\}$ is uniquely determined and $s_1 = s_2 = \cdots = s_g = 0$, $s_{g+1} = S$, and $s_{g+2} = s_{g+3} = \cdots = s_n = 0$. So we see $\underline{s_G} = \overline{s_G} = \max\{[(S+g)p_G]_* + 1 - g, 0\} = \max\{[(S+g)p_G] + 1 - g, 0\}$. Therefore, they are the best bounds if $(S+g)p_G + 1 - g < 0$. □

Rigorous statement of Theorem 3 in general cases. Generally, versions of $\{s_j\}$ and $\{s'_j\}$ are randomly chosen. So we can regard s_j and s'_j as the realizations of random variables $\tilde{s}_j$ and $\tilde{s}'_j$ (say), respectively. Here I use a tilde to signify a random variable (see footnote 13). For any versions $\{s_j\}$ and $\{s'_j\}$,

$$P[\{\tilde{s}_j\} = \{s_j\}] = 1/(\text{the number of versions of } \{s_j\}), \tag{6}$$
$$P[\{\tilde{s}'_j\} = \{s'_j\}] = 1/(\text{the number of versions of } \{s'_j\}) \tag{7}$$

hold, but the joint probability distribution of $\tilde{s}_j$ and $\tilde{s}'_j$ is not assigned. A rigorous statement of Theorem 3 in general cases is as follows: By assigning an adequate joint probability distribution of $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$ and $\tilde{s}'_j$ $(j = G, g+1, g+2, \ldots, n)$ together that does not contradict (6) nor (7), the following assertion holds:

$$P[\tilde{s}_G \leq \tilde{s}'_G \leq \tilde{s}_G + g - 1 \text{ and } \tilde{s}_j - g + 1 \leq \tilde{s}'_j \leq \tilde{s}_j \ (j = g+1,\ g+2,\ \ldots,\ n)] = 1. \tag{8}$$

I shall explain this by giving an example. Assume that $S = 2$, $n = 5$, $v_1 = v_2 = 1$, $v_3 = 4$, and $v_4 = v_5 = 3$. Before the merger, 2 versions of seats exist. One is given by $s_3 = s_4 = 1$ and $s_1 = s_2 = s_5 = 0$, while the other is given by $\dot{s}_3 = \dot{s}_5 = 1$ and $\dot{s}_1 = \dot{s}_2 = \dot{s}_4 = 0$. Let $G = \{1, 2\}$, then after the merger, also 2 versions of seats exist. One is given by $s'_3 = s'_4 = 1$ and $s'_G = s'_5 = 0$, while the other is given by $\dot{s}'_3 = \dot{s}'_5 = 1$ and $\dot{s}'_G = \dot{s}'_4 = 0$. If we carry out randomization to choose a version of seats before and after the merger independently, then $\{s_j\}$ and $\{\dot{s}'_j\}$ are chosen with probability 1/4. Here, an inequality in (4) is not satisfied for $j = 5$. So the assertion (8) does not hold for this randomization. However, after we carry out randomization to choose a version of seats before the merger (or 'carry out $\{\tilde{s}_j\}$' for short), we define $\{\tilde{s}'_j\}$ by $\tilde{s}'_j := \tilde{s}_j$ $(j = G, 3, 4, \ldots, n)$. That is, if we take a version $\{s_j\}$ before the merger, then we take a version $\{s'_j\}$ after the merger, while if we take a version $\{\dot{s}_j\}$ before the merger, then we take a version $\{\dot{s}'_j\}$ after the merger. Then we need only consider the combination of $\{s_j\}$ with $\{s'_j\}$, and $\{\dot{s}_j\}$ with $\{\dot{s}'_j\}$. Then, the inequalities (4) are satisfied and the assertion (8) follows.

On choosing a version of $\{s_j\}$. Let n_0 be the number of j's satisfying $\overline{s_j} = \underline{s_j} + 1$ $(j = 1, 2, \ldots, n)$, and $n_1 := \sum_{j=1}^{n} \overline{s_j} - S$. If $n_1 \neq 0$, then $\{s_j\}$ is not uniquely determined, so we randomly choose a version. To carry this out, we prepare n_0 cards. Each of them is written P_j where j satisfies $\overline{s_j} = \underline{s_j} + 1$. We choose n_1 cards from them. The parties chosen win only $\overline{s_j} - 1$ seats, while others win $\overline{s_j}$ seats.[22] I use a prime to signify the case after the merger. For example, $n'_1 := \overline{s'_G} + \sum_{j=3}^{n} \overline{s'_j} - S$.

[22] It is more natural to choose parties that win $\underline{s_j} + 1$ seats, while others win $\underline{s_j}$ seats. However, I do the contrary for a mathematical reason. In the natural way, a proof of the inequalities (4) under (II, iii, c) becomes complicated.

Proof of Theorem 3 (in general cases). It is clear if $g = 1$. If we show that the inequalities (4) hold when $g = 2$, we see that they hold for all g by induction. So we shall show them for $g = 2$. It is clear if $v_1 v_2 = 0$, so we may assume that $v_1 v_2 > 0$. If $v_{j_1} = 0$ for some $j_1 = 3, 4, \ldots, n$, then we may consider that the party P_{j_1} does not run. So we may assume that $v_j > 0$ for all j. We may also assume that $r = 1$ (see footnote 5). Then $v_j = s_j + \theta_j$, $\sum_{j=1}^n s_j = S$, $0 \le \theta_j \le 1$ $(j = 1, 2, \ldots, n)$, and $\prod_{j=1}^n \theta_j = 0$ hold. Denote $\theta_G := \theta_1 + \theta_2$. We divide cases as follows:

(I) $\{s_j\}$ is uniquely determined. (II) $\{s_j\}$ is not uniquely determined.

Assume (I), then $0 \le \theta_j < 1$ $(j = 1, 2, \ldots, n)$. We further divide cases as follows:

(I, i) $\theta_G < 1$. (I, ii) $\theta_G \ge 1$.

Assume (I, i), then $v_G = s_G + \theta_G$, $v_j = s_j + \theta_j$ $(j = 3, 4, \ldots, n)$, $s_G + \sum_{j=3}^n s_j = S$, $0 \le \theta_G < 1$, and $0 \le \theta_j < 1$ $(j = 3, 4, \ldots, n)$ are satisfied. There is not $j_1 = G, 3, 4, \ldots, n$ satisfying $\theta_{j_1} = 1$. Letting $r_* = 1$ and $\eta_j = \theta_j$ $(j = G, 3, 4, \ldots, n)$, we see that $\{s'_j\}$ is uniquely determined and $s'_j = s_j$ $(j = G, 3, 4, \ldots, n)$. Therefore, the inequalities (4) hold.

Assume (I, ii), then $\prod_{j=3}^n \theta_j = 0$ holds. It is enough to consider the case that $\theta_3 = \theta_4 = \cdots = \theta_{n_*+2} = 0$ and $0 < \theta_j < 1$ $(j = n_* + 3,\ n_* + 4,\ \ldots,\ n)$. We divide cases again as follows:

(I, ii, a) $\theta_G = 1$. (I, ii, b) $\theta_G > 1$.

Assume (I, ii, a), then $\theta_G = 1$, $\theta_3 = \theta_4 = \cdots = \theta_{n_*+2} = 0$, and $0 < \theta_j < 1$ $(j = n_* + 3,\ n_* + 4,\ \ldots,\ n)$ are satisfied. Hence $\{s'_j\}$ is not uniquely determined. One version is given by $s'_j = s_j$ $(j = G, 3, 4, \ldots, n)$. For this version, the inequalities (4) hold. Any other version is expressed as follows: Fix $j_0 (= 3,\ 4,\ \ldots,\ n_* + 2)$. Let $\dot{s}'_G = s_G + 1$, $\dot{s}'_{j_0} = s_{j_0} - 1$, and $\dot{s}'_j = s_j$ $(j \ne G, j_0)$. Also, for this version, the inequalities (4) hold.

Assume (I, ii, b), then $v_G = (s_G + 1) + (\theta_G - 1)$, $v_j = s_j + \theta_j$ $(j = 3, 4, \ldots, n)$, $(s_G + 1) + \sum_{j=3}^n s_j = S + 1$, $0 < \theta_G - 1 < 1$, $\theta_3 = \theta_4 = \cdots = \theta_{n_*+2} = 0$, and $0 < \theta_j < 1$ $(j = n_* + 3,\ n_* + 4,\ \ldots,\ n)$ are satisfied. Let $s''_G = s_G + 1$ and $s''_j = s_j$ $(j = 3, 4, \ldots, n)$. Then $\{s''_j\}$ is the unique sequence of the numbers of seats when we select $S + 1$ members after the merger. When we select S members, since the d'Hondt system is free from the paradox of Alabama (see Rejoinder 3 in Appendix A), we need only defeat one candidate in a party P_{j_0} where we fix j_0 such that v_{j_0}/s''_{j_0} is the smallest [i.e., $\theta_{j_0} - 1 = 0$ $(j_0 = G)$, $\theta_{j_0} = 0$ $(j_0 = 3,\ 4,\ \ldots,\ n_* + 2)$, but the former contradicts $0 < \theta_G - 1 < 1$, so the latter is satisfied]. Hence $s'_G = s''_G = s_G + 1$, and $s_j - 1 = s''_j - 1 \le s'_j \le s''_j = s_j$ $(j = 3, 4, \ldots, n)$, so the inequalities (4) hold.

Next, assume (II). We further divide cases as follows:

(II, i) $s_3,\ s_4,\ \ldots,\ s_n$ are uniquely determined. (II, ii) s_1 and s_2 are uniquely determined. (II, iii) Otherwise.

Assume (II, i), then $\{s_j\}$ has 2 versions. One of them satisfies $\theta_1 = 0$ and $\theta_2 = 1$, while the other satisfies $\theta_1 = 1$ and $\theta_2 = 0$. So s_G is uniquely determined and $\theta_G = 1$. In addition, $0 < \theta_j < 1$ $(j = 3, 4, \ldots, n)$ holds. There is not $j_0 = G, 3, 4, \ldots, n$ satisfying $\theta_{j_0} = 0$. Letting $r_* = 1$ and $\eta_j = \theta_j$ $(j = G, 3, 4, \ldots, n)$, we see that $\{s'_j\}$ is uniquely determined and $s'_j = s_j$ $(j = G, 3, 4, \ldots, n)$. Therefore, the inequalities (4) hold.

Assume (II, ii), then $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ hold. It is enough to consider the case that s_j is not uniquely determined if $j = 3,\ 4,\ \ldots,\ n_0 + 2$ while it is uniquely determined if $j = 1,\ 2,\ n_0 + 3,\ n_0 + 4,\ \ldots,\ n$. To carry out $\{\tilde{s}_j\}$, we choose $n_1 := \sum_{j=1}^n \overline{s_j} - S$ cards from n_0 cards $\mathrm{P}_3,\ \mathrm{P}_4,\ \ldots,\ \mathrm{P}_{n_0+2}$. We divide cases again as follows:

(II, ii, a) $0 < \theta_G < 1$. (II, ii, b) $\theta_G > 1$. (II, ii, c) $\theta_G = 1$.

Assume (II, ii, a). From here, we need the rigorous statement of the inequalities (4). For a fixed version $\{s_j\}$, we have $0 < \theta_G < 1$, $\theta_j \in \{0, 1\}$ $(j = 3,\ 4,\ \ldots,\ n_0 + 2)$, and $0 < \theta_j < 1$ $(j = n_0 + 3,\ n_0 + 4,\ \ldots,\ n)$. Hence $s'_G = s_G$ (uniquely determined), $\overline{s'_j} = \overline{s_j} = \underline{s'_j} + 1 = \underline{s_j} + 1$ $(j = 3,\ 4,\ \ldots,\ n_0 + 2)$, and $s'_j = s_j$ $(j = n_0 + 3,\ n_0 + 4,\ \ldots,\ n$, uniquely determined). To carry out $\{\tilde{s}'_j\}$, we choose $n'_1 = n_1$ cards from $n'_0 = n_0$ cards $\mathrm{P}_3,\ \mathrm{P}_4,\ \ldots,\ \mathrm{P}_{n_0+2}$. If we carry out this randomization and that of $\{\tilde{s}_j\}$ independently, then the statement (8) does not hold. However, after carrying out $\{\tilde{s}_j\}$, define $\{\tilde{s}'_j\}$ from $\{\tilde{s}_j\}$ by $\tilde{s}'_j = \tilde{s}_j$ $(j = G, 3, 4, \ldots, n)$. Then this $\{\tilde{s}'_j\}$ satisfies (7) and (8).

Assume (II, ii, b). For a fixed version $\{s_j\}$, we have $v_G = (s_G + 1) + (\theta_G - 1)$, $v_j = s_j + \theta_j$ $(j = 3, 4, \ldots, n)$, $(s_G + 1) + \sum_{j=3}^n s_j = S + 1$, $0 < \theta_G - 1 < 1$, $\theta_j \in \{0, 1\}$ $(j = 3,\ 4,\ \ldots,\ n_0 + 2)$, and $0 < \theta_j < 1$ $(j = n_0 + 3,\ n_0 + 4,\ \ldots,\ n)$. Let $s''_G = s_G + 1$ and $s''_j = s_j$ $(j = 3, 4, \ldots, n)$. Then $\{s''_j\}$ is a sequence of the numbers of seats when we select $S + 1$ members after the merger. When we select S members, we need only defeat one candidate in a party P_{j_0} where we fix j_0 such that $\theta_{j_0} = 0$ ($j_0 = 3, 4, \ldots, n_0+2$). If j_0 is uniquely determined, then so is $\{s'_j\}$ and the inequalities (4) follow. Assume that j_0 is not uniquely determined. Then $s'_G = s_G + 1$ (uniquely determined), $\overline{s'_j} = \overline{s_j} = \underline{s'_j} + 1 = \underline{s_j} + 1$ ($j = 3, 4, \ldots, n_0+2$), and $s'_j = s_j$ ($j = n_0+3, n_0+4, \ldots, n$, uniquely determined). To carry out $\{\tilde{s}'_j\}$, we choose $n'_1 = n_1 + 1$ cards from $n'_0 = n_0$ cards $P_3, P_4, \ldots, P_{n_0+2}$. After carrying out $\{\tilde{s}_j\}$, if we choose one more card (say $P_{\tilde{j}_0}$) and let $\tilde{s}'_G := \tilde{s}_G + 1$, $\tilde{s}'_{\tilde{j}_0} := \tilde{s}_{\tilde{j}_0} - 1$, and $\tilde{s}'_j := \tilde{s}_j$ ($j \neq G, \tilde{j}_0$), then this $\{\tilde{s}'_j\}$ clearly satisfies (8). It also satisfies (7) because choosing n'_1 cards at once is equivalent to choosing $n'_1 - 1$ cards and adding one more card.

Assume (II, ii, c). For a fixed version $\{s_j\}$, we have $\theta_G = 1$, $\theta_j \in \{0,1\}$ ($j = 3, 4, \ldots, n_0+2$), and $0 < \theta_j < 1$ ($j = n_0+3, n_0+4, \ldots, n$). Hence $\overline{s'_G} = s_G + 1$, $\underline{s'_G} = s_G$ (s_G is uniquely determined while s'_G is not), $\overline{s'_j} = \overline{s_j} = \underline{s'_j} + 1 = \underline{s_j} + 1$ ($j = 3, 4, \ldots, n_0+2$), and $s'_j = s_j$ ($j = n_0+3, n_0+4, \ldots, n$, uniquely determined). To carry out $\{\tilde{s}'_j\}$, we choose $n'_1 = n_1 + 1$ cards from $n'_0 = n_0 + 1$ cards $P_G, P_3, P_4, \ldots, P_{n_0+2}$. After carrying out $\{\tilde{s}_j\}$, we add the card P_G with probability n'_1/n'_0, while we choose another card (say $P_{\tilde{j}_0}$) from cards $P_3, P_4, \ldots, P_{n_0+2}$ except already chosen cards with probability $1 - (n'_1/n'_0)$. If the card P_G is added, then let $\tilde{s}'_j := \tilde{s}_j$ ($j = G, 3, 4, \ldots, n$). If the card P_G is not added, then let $\tilde{s}'_G := \tilde{s}_G + 1$, $\tilde{s}'_{\tilde{j}_0} := \tilde{s}_{\tilde{j}_0} - 1$, and $\tilde{s}'_j := \tilde{s}_j$ ($j \neq G, \tilde{j}_0$). Then this $\{\tilde{s}'_j\}$ clearly satisfies (8). It also satisfies (7) because choosing n'_1 cards from n'_0 cards $P_G, P_3, P_4, \ldots, P_{n_0+2}$ at once is equivalent to the following: choose $n'_1 - 1$ cards from $n'_0 - 1$ cards $P_3, P_4, \ldots, P_{n_0+2}$ and add the card P_G with probability n'_1/n'_0 while choosing another card from cards $P_3, P_4, \ldots, P_{n_0+2}$ except already chosen cards with probability $1 - (n'_1/n'_0)$.

Assume (II, iii). We further divide cases as follows:

(II, iii, a) Either s_1 or s_2 is uniquely determined. (II, iii, b) Neither s_1 nor s_2 is uniquely determined.

Assume (II, iii, a), then it is enough to consider the case that s_j is not uniquely determined if $j = 1, 3, 4, \ldots, n_0 + 1$ while it is uniquely determined if $j = 2, n_0+2, n_0+3, \ldots, n$. For a fixed version $\{s_j\}$, we have $\theta_1 = 0$ or $\theta_1 = 1$. In addition, $0 < \theta_2 < 1$ holds. If $\theta_1 = 0$, then we have $0 < \theta_G < 1$, $\theta_j \in \{0,1\}$ ($j = 3, 4, \ldots, n_0+1$), and $0 < \theta_j < 1$ ($j = n_0+2, n_0+3, \ldots, n$). If $n_0 = 2$, then $\{s'_j\}$ is uniquely determined and $s'_j = s_j$ ($j = G, 3, 4, \ldots, n$). For the other version $\{\dot{s}_j\}$, which satisfies $\theta_1 = 1$, we have $s'_G = s_G = \dot{s}_G + 1$ and '$s'_j = s_j = \dot{s}_j$ or $s'_j = s_j = \dot{s}_j - 1$" ($j = 3, 4, \ldots, n$). So this version also satisfies the inequalities (4). Assume that $n_0 \geq 3$, then $\{s'_j\}$ is not uniquely determined. We get $s'_G = \overline{s_G}$ (s'_G is uniquely determined while s_G is not), $\overline{s'_j} = \overline{s_j} = \underline{s'_j} + 1 = \underline{s_j} + 1$ ($j = 3, 4, \ldots, n_0+1$), and $s'_j = s_j$ ($j = n_0+2, n_0+3, \ldots, n$, uniquely determined). To carry out $\{\tilde{s}'_j\}$, we choose $n'_1 = n_1$ cards from $n'_0 = n_0 - 1$ cards $P_3, P_4, \ldots, P_{n_0+1}$. After carrying out $\{\tilde{s}_j\}$, if the card P_1 is not chosen, then let $\tilde{s}'_j := \tilde{s}_j$ ($j = G, 3, 4, \ldots, n$). If the card P_1 is chosen, then neglect it and choose one more card (say $P_{\tilde{j}_0}$) and let $\tilde{s}'_G := \tilde{s}_G + 1$, $\tilde{s}'_{\tilde{j}_0} := \tilde{s}_{\tilde{j}_0} - 1$, and $\tilde{s}'_j := \tilde{s}'_j$ ($j \neq G, \tilde{j}_0$). Then this $\{\tilde{s}'_j\}$ clearly satisfies (8). It also satisfies (7) because choosing n'_1 cards from n'_0 cards $P_3, P_4, \ldots, P_{n'_0+2}$ at once is equivalent to the following: choose n'_1 cards from $n'_0 + 1$ cards $P_1, P_3, P_4, \ldots, P_{n'_0+2}$ and if P_1 is chosen, then neglect it and choose one more card.

Assume (II, iii, b), then it is enough to consider the case that s_j is not uniquely determined if $j = 1, 2, 3, 4, \ldots, n_0$ while it is uniquely determined if $j = n_0+1, n_0+2, \ldots, n$. For a fixed version $\{s_j\}$, we have $(\theta_1, \theta_2) = (0,0), (0,1), (1,0), (1,1)$. Note that $\overline{s_G} = \underline{s_G} + 2$ in (and only in) this case. If $\theta_1 = \theta_2 = 0$, then $\theta_G = 0$, $\theta_j \in \{0,1\}$ ($j = 3, 4, \ldots, n_0$), and $0 < \theta_j < 1$ ($j = n_0+1, n_0+2, \ldots, n$) are satisfied. Hence $\overline{s'_G} = \overline{s_G} = \underline{s'_G} + 1 = \underline{s_G} + 2$, $\overline{s'_j} = \overline{s_j} = \underline{s'_j} + 1 = \underline{s_j} + 1$ ($j = 3, 4, \ldots, n_0$), $s'_j = s_j$ ($j = n_0+1, n_0+2, \ldots, n$, uniquely determined). To carry out $\{\tilde{s}'_j\}$, we choose $n'_1 = n_1$ cards from $n'_0 = n_0 - 1$ cards $P_G, P_3, P_4, \ldots, P_{n_0}$. After carrying out $\{\tilde{s}_j\}$, regard the card P_2 as the card P_G. If the card P_1 is not chosen, then let $\tilde{s}'_j := \tilde{s}_j$ ($j = G, 3, 4, \ldots, n$). If the card P_1 is chosen, then neglect it and choose one more card (say $P_{\tilde{j}_0}$). If it is the card P_2, then regard it as the card P_G and let $\tilde{s}'_j := \tilde{s}_j$ ($j = G, 3, 4, \ldots, n$). Otherwise, let $\tilde{s}'_G := \tilde{s}_G + 1$, $\tilde{s}'_{\tilde{j}_0} := \tilde{s}_{\tilde{j}_0} - 1$, and $\tilde{s}'_j := \tilde{s}'_j$ ($j \neq G, \tilde{j}_0$). Then this $\{\tilde{s}'_j\}$ clearly satisfies (8). It also satisfies (7) because choosing n'_1 cards from n'_0 cards $P_G, P_3, P_4, \ldots, P_{n'_0+1}$ at once is equivalent to the following: choose n'_1 cards from $n'_0 + 1$ cards $P_1, P_2, P_3, P_4, \ldots, P_{n'_0+1}$, regard P_2 as P_G, and if P_1 is chosen, then neglect it and choose one more card.

Hence we have completed the proof in all the cases. $\square$

Proof of Lemma 2.[23] It is enough to derive the approximation for $j = 1$. By Assumption 5, consider that S is also the realization of a random variable $\tilde{S}$. Then we have

$$P[\tilde{\jmath}_0 = j_0|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)] \approx \begin{cases} p_{j_0}/p_M & \text{for } j_0 \in M, \\ 0 & \text{otherwise,} \end{cases}$$

where $M = \{j : p_j \geq t\}$ for some small $t > 0$. To see this,

$$\begin{aligned}
&P[\tilde{\jmath}_0 = j_0|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)] \\
=&P[\text{The } \tilde{S}\text{th largest value of } \{p_j/l\}_{j=1,2,\dots,n}^{l=1,2,\dots} \text{ is attained for } j = j_0.\ ] \\
=&\frac{[\text{The number of } S\text{'s } (\underline{S} \leq S \leq \overline{S}) \text{ where the } S\text{th largest value of } \{p_j/l\} \text{ is attained for } j = j_0\,]}{\overline{S} - \underline{S} + 1},
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{P[\tilde{\jmath}_0 = 1|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)]}{P[\tilde{\jmath}_0 = 2|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)]} \\
=&\frac{[\text{The number of } S\text{'s } (\underline{S} \leq S \leq \overline{S}) \text{ where the } S\text{th largest value of } \{p_j/l\} \text{ is attained for } j = 1\,]}{[\text{The number of } S\text{'s } (\underline{S} \leq S \leq \overline{S}) \text{ where the } S\text{th largest value of } \{p_j/l\} \text{ is attained for } j = 2\,]} \\
\approx&\frac{p_1}{p_2} \quad \text{if neither } p_1 \text{ nor } p_2 \text{ is so small.}
\end{aligned}$$

The suffixes 1 and 2 above are nonessential. Therefore, for some constant a,

$$P[\tilde{\jmath}_0 = j_0|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)] \approx a p_{j_0}$$

holds if p_{j_0} is not so small (say $p_{j_0} \geq t > 0$). Otherwise, it is approximately (or exactly) 0. Since $\sum_{j_0=1}^{n} P[\tilde{\jmath}_0 = j_0|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)] = 1$, we get $a \approx 1/p_M$.

By Assumption 1, we may consider

$$P[\tilde{\jmath}_0 = j_0|\tilde{p}_j = p_j\ (j = 1, 2, \dots, n)] \approx \begin{cases} p^*_{j_0}/p^*_{M^*} = p^{**}_{j_0} & \text{for } j_0 \in M^*, \\ 0 & \text{otherwise,} \end{cases}$$

where $M^* = \{j : p^*_j \geq t^*\}$ for some small $t^* > 0$. The right-hand side is independent of p_j, so we have

$$P[\tilde{\jmath}_0 = j_0] \approx \begin{cases} p^{**}_{j_0} & \text{for } j_0 \in M^*, \\ 0 & \text{otherwise,} \end{cases} \quad \text{where } \tilde{S} \text{ is a random variable.}$$

By Assumption 5, we get

$$P[\tilde{\jmath}_0 = j_0|\tilde{S} = S] \approx \begin{cases} p^{**}_{j_0} & \text{for } j_0 \in M^*, \\ 0 & \text{otherwise.} \end{cases}$$

I consider that S is a constant again. So this should be rewritten by

$$P[\tilde{\jmath}_0 = j_0] \approx \begin{cases} p^{**}_{j_0} & \text{for } j_0 \in M^*, \\ 0 & \text{otherwise.} \end{cases}$$

[23] In the following proof, only Assumptions 1 ("only a little"), 5, and 6 are used explicitly. In Assumption 1, "not too little" is not explicitly used, but if $\tilde{p}_j$ can change too little, then it contradicts Assumption 6. In Assumption 2, for $S + n \ll \tilde{V}$, see the footnote of Assumption 6. In Assumption 2, if $\tilde{V} \not\approx \overline{V}$ is not satisfied, then $v_i + r - a > \overline{V}$ might hold and Assumption 6 does not make sense. If Assumption 3 does not hold, then r becomes too large and Assumption 6 contradicts Assumption 1 ("only a little"). If it were not for Assumption 4, then Assumption 5 would not make sense.

Next, assume that $1 \in M^*$. When v_j's $(j = 1, 2, \ldots, n)$ are given, we can express $v_j = r(s_j + \theta_j)$ $(j = 1, 2, \ldots, n)$. Assume that $\theta_1 \neq 0$. Regard $\tilde{v}_j = v_j$ $(j = 2, 3, \ldots, n)$ as a constant and consider that only $\tilde{v}_1$ can change its value. Letting $a = r\theta_1$, by Assumption 6, we see that the approximate conditional distribution of $\tilde{v}_1$ under the conditions $rs_1 < \tilde{v}_1 < r(s_1 + 1)$ and $\tilde{v}_j = v_j$ $(j = 2, 3, \ldots, n)$ is the uniform distribution on the interval $(rs_1, r(s_1 + 1))$.[24] This is equivalent to say that the conditional distribution of $\tilde{\theta}_1$ under the conditions $\tilde{\theta}_1 \neq 0$, $\tilde{s}_1 = s_1$, and $\tilde{v}_j = v_j$ $(j = 2, 3, \ldots, n)$ is approximately the uniform distribution on the interval $(0, 1)$, which is independent of s_1 and v_j $(j = 2, 3, \ldots, n)$. Hence the conditional probability distribution of $\tilde{\theta}_1$ under the condition $\tilde{\theta}_1 \neq 0$ is approximately the uniform distribution on the interval $(0, 1)$. Therefore, we have

$$\theta_1^* = 0 \cdot P[\tilde{\theta}_1 = 0] + E(\tilde{\theta}_1 | \tilde{\theta}_1 \neq 0) P[\tilde{\theta}_1 \neq 0] \approx \frac{1 - p_1^{**}}{2}. \qquad \square$$

Reason the standpoint based on u_j^* does not justify the probabilistic approach. In this standpoint, $\tilde{v}_j$ is distributed as $B(\overline{V}, u_j^*)$, so $E(\tilde{v}_j) = \overline{V} u_j^*$ and $\mathrm{Var}(\tilde{v}_j) = \overline{V} u_j^*(1 - u_j^*) \leq \overline{V}/4$. Fix $v_j \approx \overline{V} u_j^*$ $(j = 1, 2, \ldots, n)$, then V and r are determined correspondingly, and we can consider that so is $\{s_j\}$. Since $\overline{V}$ is sufficiently large, by the central limit theorem, $\tilde{v}_j$ is approximately distributed as a normal distribution and $P\left[|\tilde{v}_j - \overline{V} u_j^*| < \sqrt{\overline{V}}\right] > 0.95$ holds. However, in the discussion above, we have seen that the conditional probability distribution of $\tilde{v}_1$ under some conditions is approximately the uniform distribution on the interval $(rs_1, r(s_1 + 1))$. If the two approximations do not contradict, $r \ll 2\sqrt{\overline{V}}$ should hold. Since $V/(S + n - 1) \leq r$, we have $V/(S + n - 1) \ll 2\sqrt{\overline{V}}$. By approximating $V \approx \overline{V}(1 - u_0^*)$, we get $\overline{V}(1 - u_0^*)/(S + n - 1) \ll 2\sqrt{\overline{V}}$, so $(1 - u_0^*)\sqrt{\overline{V}} \ll 2(S + n - 1)$, hence $(1 - u_0^*)^2 \overline{V} \ll 4(S + n - 1)^2$. Since we may consider $V \approx (1 - u_0^*)\overline{V}$, we get $(1 - u_0^*)V \ll 4(S + n - 1)^2$. Actually, V is very much larger than $S + n$, and $1 - u_0^*$ is not so small, so this does not hold. For the actual data, $V > 10^7$, $1 - u_0^* > 0.5$, so $(1 - u_0^*)V > 5 \times 10^6$ in all blocs, and $4(S + n - 1)^2 < 2 \times 10^5$ even for the constituency covering the whole nation. Note that even if we suppose that u_j^* depends on each individual, as long as we assume that each one *independently* votes (or abstains), the two approximations give a contradiction. $\square$

Explanation for letting $\widehat{M^*} := \{j : s_j + \theta_j \geq 1/2\}$. For a fixed j, consider whether we should let $j \in \widehat{M^*}$ or not. Remember $v_j = r(s_j + \theta_j)$, and that, if $j \in M^*$, then $\theta_j^* \approx (1 - p_j^{**})/2$. If $s_j + \theta_j$ is very small, this expectation is inadequate. For example, for the actual data in the Kinki bloc, $s_j + \theta_j \approx 0.08$ (very small) for the DRL. Under this expectation, however, this is due to bad luck, not due to few electors' support. So we should let $j \notin \widehat{M^*}$ in such a case. We need a borderline of $s_j + \theta_j$. Though this is for convenience' sake, I make a borderline as follows: Since $\theta_j^* \approx 1/2$ if p_j^* is small but $j \in M^*$, I let a borderline be $1/2$, that is, $\widehat{M^*} := \{j : s_j + \theta_j \geq 1/2\}$, and define

$$\widehat{\theta_j^*} := \begin{cases} \dfrac{1 - \widehat{p_j^{**}}}{2} & \text{for } j \in \widehat{M^*}, \\ \theta_j & \text{for } j \notin \widehat{M^*}, \end{cases}$$

where $\widehat{p_j^{**}} := \widehat{p_j^*} / \sum_{i \in \widehat{M^*}} \widehat{p_i^*}$ for $j \in \widehat{M^*}$. $\square$

[24] Since $\theta_1 \neq 0$ is assumed, $s_1 \neq S$ is satisfied.

Appendix C

The following tables show detailed actual and *falsified* data. **Hokkaido, Tohoku, K.Kanto** (Kita-Kanto), **M.Kanto** (Minami-Kanto), **Tokyo, H.Shin.** (Hokuriku-Shin'etsu), **Tokai, Kinki, Chugoku, Shikoku,** and **Kyushu,** are the names of the blocs. LDP, NFP, etc. are the abbreviations of the parties explained in Table 1. In each bloc, for each party (not a combination), the meanings of Votes, Percentage, etc. are as follows:

Votes	(v_j) The number of the votes that each party gets
Percentage	(p_j) The relative proportion of the votes that each party gets
Upper	The upper bound of the number of the seats given in the text
Seats	(s_j) The number of the seats
Lower	The lower bound of the number of the seats given in the text
Perfect	(Sp_j) The number of the perfect PR seats
Estimate	($\widehat{s_j^*}$) See the text.
θ_j	See the text.

We can get data of Votes, Percentage, Seats by usual Japanese newspapers on October 21 (evening papers) or 22, 1996. I used the data of Votes and computed others by *Mathematica* for Macintosh. For combinations of parties, the meanings of Votes, Percentage, etc. are as follows:

Votes	(v_G) The number of the votes that each combination of the parties altogether gets
Percentage	(p_j) The relative proportion of the votes that each party gets
Upper	The upper bound of the number of the seats given in the text
Seats	(s_G) The number of the seats
Lower	The lower bound of the number of the seats given in the text
Perfect	(Sp_G) The number of the perfect PR seats
Estimate	($\widehat{s_G^*}$) See the text.
θ_G	See the text.

For **Total**, I give the summation of Votes, Upper, Seats, Lower, Estimates, and θ_j (θ_G) with respect to the blocs. Of course Percentage is not the summation. It is based on the total votes. I have to explain the meaning of Perfect in **Total**. It is not the summation of Perfect with respect to the blocs, but it is $S^{(\mathrm{N})}p_j^{(\mathrm{N})}$, that is, it is based on the total votes and the total seats. I give the value of the total of Perfect, that is, $\sum_{k=1}^{b} S^{(k)}p_j^{(k)}$, as a reference below it. The reason is as follows: It is important to compare s_j or $\widehat{s_j^*}$ with $S^{(\mathrm{N})}p_j^{(\mathrm{N})}$, not with $\sum_{k=1}^{b} S^{(k)}p_j^{(k)}$. **Nation** means the constituency covering the whole nation. Here I calculate supposing the PR were carried out under the constituency covering the whole nation.

TABLE 4. Detailed actual data

Hokkaido	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	740677	552847	835072	396923	0	100807	0	0	0	2626326
Percentage	28.2	21.05	31.8	15.11	0	3.84	0	0	0	100
Upper	3	2	4	1	0	0	0	0	0	10
Seats	3	2	3	1	0	0	0	0	0	9
Lower	2	2	3	1	0	0	0	0	0	8
Perfect	2.54	1.89	2.86	1.36	0	0.35	0	0	0	9
Estimates	2.72	1.91	3.13	1.23	0	0.01	0	0	0	9
θ_j	0	0.24	0.38	0.61	0	0.41	0	0	0	1.64

Tohoku	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1630777	1532987	513410	442790	382271	84167	0	37661	0	4624063
Percentage	35.27	33.15	11.1	9.58	8.27	1.82	0	0.81	0	100
Upper	7	7	2	2	1	0	0	0	0	19
Seats	6	6	2	1	1	0	0	0	0	16
Lower	5	5	1	1	1	0	0	0	0	13
Perfect	5.64	5.3	1.78	1.53	1.32	0.29	0	0.13	0	16
Estimates	6.2	5.8	1.61	1.32	1.07	0.01	0	0	0	16
θ_j	0.38	0	0.01	0.73	0.5	0.33	0	0.15	0	2.1

K.Kanto	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1962854	1500349	965328	722792	282201	81836	64350	47020	0	5626730
Percentage	34.88	26.66	17.16	12.85	5.02	1.45	1.14	0.84	0	100
Upper	9	7	4	3	1	0	0	0	0	24
Seats	8	6	4	2	1	0	0	0	0	21
Lower	7	5	3	2	1	0	0	0	0	18
Perfect	7.33	5.6	3.6	2.7	1.05	0.31	0.24	0.18	0	21
Estimates	7.98	5.98	3.67	2.62	0.72	0.01	0.01	0	0	21
θ_j	0.13	0.22	0	0.995	0.17	0.34	0.27	0.19	0	2.32

M.Kanto	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1820846	1667552	1331850	881751	403875	102906	0	71756	0	6280536
Percentage	28.99	26.55	21.21	14.04	6.43	1.64	0	1.14	0	100
Upper	8	7	6	4	1	0	0	0	0	26
Seats	7	7	5	3	1	0	0	0	0	23
Lower	6	6	5	3	1	0	0	0	0	21
Perfect	6.67	6.11	4.88	3.23	1.48	0.38	0	0.26	0	23
Estimates	7.11	6.47	5.07	3.18	1.19	−0.01	0	−0.01	0	23
θ_j	0.64	0	0.59	0.7	0.7	0.43	0	0.3	0	3.36

Tokyo	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1398791	1275432	1213677	923764	280391	68260	0	25813	0	5186128
Percentage	26.97	24.59	23.4	17.81	5.41	1.32	0	0.5	0	100
Upper	6	6	5	4	1	0	0	0	0	22
Seats	5	5	5	3	1	0	0	0	0	19
Lower	5	4	4	3	1	0	0	0	0	17
Perfect	5.12	4.67	4.45	3.38	1.03	0.25	0	0.09	0	19
Estimates	5.41	4.89	4.62	3.4	0.68	0	0	0	0	19
θ_j	0.76	0.25	0	0.81	0.16	0.28	0	0.11	0	2.37

TABLE 4. Detailed actual data (Continued)

H.Shin.	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1407828	1180904	494666	387664	243287	57643	125694	0	0	3897686
Percentage	36.12	30.3	12.69	9.95	6.24	1.48	3.22	0	0	100
Upper	6	5	2	1	1	0	0	0	0	15
Seats	5	4	2	1	1	0	0	0	0	13
Lower	5	4	1	1	0	0	0	0	0	11
Perfect	4.7	3.94	1.65	1.29	0.81	0.19	0.42	0	0	13
Estimates	5.37	4.42	1.56	1.12	0.51	0	0.02	0	0	13
θ_j	0.79	0.85	0.03	0.59	0	0.24	0.52	0	0	3.02

Tokai	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2042948	2107536	955464	756037	378414	79449	0	58965	0	6378813
Percentage	32.03	33.04	14.98	11.85	5.93	1.25	0	0.92	0	100
Upper	9	9	4	3	1	0	0	0	0	26
Seats	8	8	3	3	1	0	0	0	0	23
Lower	7	7	3	2	1	0	0	0	0	20
Perfect	7.37	7.6	3.45	2.73	1.36	0.29	0	0.21	0	23
Estimates	7.85	8.11	3.4	2.59	1.05	0	0	0	0	23
θ_j	0.11	0.36	0.79	0	0.5	0.32	0	0.23	0	2.31

Kinki	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2497411	2567452	1223192	1539172	542047	122989	234849	58320	18844	8804276
Percentage	28.37	29.16	13.89	17.48	6.16	1.4	2.67	0.66	0.21	100
Upper	11	11	5	7	2	0	1	0	0	37
Seats	10	10	5	6	2	0	0	0	0	33
Lower	9	9	4	5	2	0	0	0	0	29
Perfect	9.36	9.62	4.58	5.77	2.03	0.46	0.88	0.22	0.07	33
Estimates	9.94	10.24	4.62	5.94	1.77	0.01	0.48	0	0	33
θ_j	0.21	0.49	0	0.29	0.22	0.503	0.96	0.24	0.08	2.99

Chugoku	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1578140	883319	464197	356108	234642	125824	43772	0	0	3686002
Percentage	42.81	23.96	12.59	9.66	6.37	3.41	1.19	0	0	100
Upper	8	4	2	1	1	0	0	0	0	16
Seats	6	3	2	1	1	0	0	0	0	13
Lower	5	3	1	1	0	0	0	0	0	10
Perfect	5.57	3.12	1.64	1.26	0.83	0.44	0.15	0	0	13
Estimates	6.43	3.38	1.54	1.06	0.53	0.05	0	0	0	13
θ_j	0.8	0.81	0	0.53	0.01	0.54	0.19	0	0	2.88

Shikoku	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	783589	455269	245323	227014	132868	39067	0	0	0	1883130
Percentage	41.61	24.18	13.03	12.06	7.06	2.07	0	0	0	100
Upper	4	2	1	1	0	0	0	0	0	8
Seats	3	2	1	1	0	0	0	0	0	7
Lower	3	1	1	0	0	0	0	0	0	5
Perfect	2.91	1.69	0.91	0.84	0.49	0.15	0	0	0	7
Estimates	3.53	1.84	0.76	0.67	0.18	0.02	0	0	0	7
θ_j	0.45	0.01	0.08	0	0.59	0.17	0	0	0	1.3

TABLE 4. Detailed actual data (Continued)

Kyushu	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2342094	1856406	707011	634728	667244	100523	113428	154071	0	6575505
Percentage	35.62	28.23	10.75	9.65	10.15	1.53	1.73	2.34	0	100
Upper	10	8	3	2	3	0	0	0	0	26
Seats	9	7	3	2	2	0	0	0	0	23
Lower	8	6	2	2	2	0	0	0	0	20
Perfect	8.19	6.49	2.47	2.22	2.33	0.35	0.4	0.54	0	23
Estimates	9.09	7.1	2.39	2.1	2.23	−0.02	−0.03	0.13	0	23
θ_j	0.94	0.88	0	0.69	0.83	0.43	0.48	0.65	0	4.9

Total	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	18205955	15580053	8949190	7268743	3547240	963471	582093	453606	18844	55569195
Percentage	32.76	28.04	16.1	13.08	6.38	1.73	1.05	0.82	0.03	100
Upper	81	68	38	29	12	0	1	0	0	229
Seats	70	60	35	24	11	0	0	0	0	200
Lower	62	52	28	21	9	0	0	0	0	172
Perfect	65.53	56.07	32.21	26.16	12.77	3.47	2.1	1.63	0.07	200
Estimates	71.63	60.13	32.38	25.23	9.93	0.08	0.48	0.14	0	200
$\sum_k \theta_j^{(k)}$	5.21	4.11	1.89	5.96	3.66	3.99	2.41	1.88	0.08	29.18
$\sum_k S^{(k)} p_j^{(k)}$	65.39	56.04	32.27	26.31	12.75	3.45	2.09	1.63	0.07	200

Nation	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	18205955	15580053	8949190	7268743	3547240	963471	582093	453606	18844	55569195
Percentage	32.76	28.04	16.1	13.08	6.38	1.73	1.05	0.82	0.03	100
Upper	68	58	33	27	13	3	2	1	0	205
Seats	66	57	32	26	13	3	2	1	0	200
Lower	65	56	32	26	12	3	2	1	0	197
Perfect	65.53	56.07	32.21	26.16	12.77	3.47	2.1	1.63	0.07	200
Estimates	66.36	56.72	32.36	26.19	12.53	3.04	1.64	1.17	0	200
θ_j	0.72	0.1	0.8	0.64	0	0.53	0.13	0.66	0.07	3.65

TABLE 4. Detailed actual data (Continued)

Hokkaido	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	740677	1293524	2128596	396923	100807
Percentage	28.2	49.25	81.05	15.11	3.84
Upper	3	5	8	1	0
Seats	3	5	8	1	0
Lower	2	4	7	1	0
Perfect	2.54	4.43	7.29	1.36	0.35
Estimates	2.72	4.63	7.76	1.23	0.01
θ_G	0	0.24	0.62	0.61	0.41

Tohoku	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	2013048	3163764	3677174	825061	121828
Percentage	43.53	68.42	79.52	17.84	2.63
Upper	9	14	15	3	0
Seats	7	12	14	2	0
Lower	6	11	13	2	0
Perfect	6.97	10.95	12.72	2.85	0.42
Estimates	7.27	11.99	13.6	2.39	0.01
θ_G	0.88	0.38	0.39	1.23	0.48

K.Kanto	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	2309405	3463203	4428531	1004993	193206
Percentage	41.04	61.55	78.71	17.86	3.43
Upper	10	16	20	4	0
Seats	9	14	18	3	0
Lower	7	13	16	3	0
Perfect	8.62	12.93	16.53	3.75	0.72
Estimates	8.71	13.97	17.64	3.34	0.02
θ_G	0.57	0.35	0.35	1.16	0.8

M.Kanto	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	2224721	3488398	4820248	1285626	174662
Percentage	35.42	55.54	76.75	20.47	2.78
Upper	9	15	20	5	0
Seats	8	14	19	4	0
Lower	7	12	17	4	0
Perfect	8.15	12.77	17.65	4.71	0.64
Estimates	8.3	13.58	18.64	4.37	−0.02
θ_G	1.34	0.64	1.23	1.4	0.73

Tokyo	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	1679182	2674223	3887900	1204155	94073
Percentage	32.38	51.56	74.97	23.22	1.81
Upper	7	12	17	5	0
Seats	6	10	15	4	0
Lower	5	9	14	3	0
Perfect	6.15	9.8	14.24	4.41	0.34
Estimates	6.09	10.29	14.92	4.08	0
θ_G	0.92	1.02	1.02	0.96	0.39

TABLE 4. Detailed actual data (Continued)

H.Shin.	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	1776809	2588732	3083398	630951	183337
Percentage	45.59	66.42	79.11	16.19	4.7
Upper	7	11	13	2	0
Seats	6	9	11	2	0
Lower	5	8	10	1	0
Perfect	5.93	8.63	10.28	2.1	0.61
Estimates	5.91	9.79	11.35	1.63	0.02
θ_G	1.3	1.64	1.67	0.59	0.75

Tokai	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	2421362	4150484	5105948	1134451	138414
Percentage	37.96	65.07	80.05	17.78	2.17
Upper	10	18	21	4	0
Seats	9	16	19	4	0
Lower	8	15	18	3	0
Perfect	8.73	14.97	18.41	4.09	0.5
Estimates	8.89	15.96	19.36	3.63	0.01
θ_G	0.61	0.47	1.26	0.5	0.55

Kinki	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	3274307	5064863	6288055	2081219	435002
Percentage	37.19	57.53	71.42	23.64	4.94
Upper	14	23	27	9	1
Seats	12	20	25	8	0
Lower	11	19	23	7	0
Perfect	12.27	18.98	23.57	7.8	1.63
Estimates	12.19	20.18	24.8	7.7	0.5
θ_G	1.38	0.7	0.7	0.51	1.78

Chugoku	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	1856554	2461459	2925656	590750	169596
Percentage	50.37	66.78	79.37	16.03	4.6
Upper	8	12	13	2	0
Seats	7	9	11	2	0
Lower	6	9	10	1	0
Perfect	6.55	8.68	10.32	2.08	0.6
Estimates	6.96	9.81	11.35	1.6	0.05
θ_G	1	1.61	1.61	0.55	0.73

Shikoku	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	916457	1238858	1484181	359882	39067
Percentage	48.67	65.79	78.81	19.11	2.07
Upper	5	7	7	2	0
Seats	3	5	6	1	0
Lower	3	4	5	0	0
Perfect	3.41	4.61	5.52	1.34	0.15
Estimates	3.71	5.37	6.13	0.85	0.02
θ_G	1.04	0.46	0.54	0.59	0.17

TABLE 4. Detailed actual data (Continued)

Kyushu	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	3122766	4198500	4905511	1301972	368022
Percentage	47.49	63.85	74.6	19.8	5.6
Upper	13	18	20	5	1
Seats	11	16	19	4	0
Lower	10	14	17	3	0
Perfect	10.92	14.69	17.16	4.55	1.29
Estimates	11.3	16.19	18.59	4.33	0.08
θ_G	2.25	1.82	1.82	1.52	1.56

Total	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	22335288	33786008	42735198	10815983	2018014
Percentage	40.19	60.8	76.9	19.46	3.63
Upper	95	151	181	42	2
Seats	81	130	165	35	0
Lower	70	118	150	28	0
Perfect	80.39	121.6	153.81	38.93	7.26
Estimates	82.05	131.76	164.14	35.16	0.7
$\sum_k \theta_G^{(k)}$	11.29	9.32	11.21	9.62	8.35
$\sum_k S^{(k)} p_G^{(k)}$	80.23	121.43	153.7	39.06	7.24

Nation	LDP+SDP+NPS	LDP+NFP	LDP+NFP+MIN	JCP+SDP	NSP+NPS+JR+DRL
Votes	22335288	33786008	42735198	10815983	2018014
Percentage	40.19	60.8	76.9	19.46	3.63
Upper	82	125	158	40	7
Seats	81	123	155	39	6
Lower	79	121	154	38	4
Perfect	80.39	121.6	153.81	38.93	7.26
Estimates	80.52	123.07	155.44	38.72	5.84
θ_G	0.85	0.82	1.62	0.64	1.4

	S	n	V/S	r	$V/(S+n-1)$	$V/(S+1)$	$r\Theta$
Hokkaido	9	5	291814	246892.3	202025.1	262632.6	404295
Tohoku	16	7	289003.9	255497.8	210184.7	272003.7	536097.7
K.Kanto	21	8	267939.5	241332	200954.6	255760.5	558758
M.Kanto	23	7	273066.8	238221.7	216570.2	261689	801436.6
Tokyo	19	7	272954.1	242735.4	207445.1	259306.4	574155.4
H.Shin.	13	7	299822	243287	205141.4	278406.1	734955
Tokai	23	7	277339.7	252012.3	219959.1	265783.9	582529.3
Kinki	33	9	266796.2	244638.4	214738.4	258949.3	731208.8
Chugoku	13	7	283538.6	232098.5	194000.1	263285.9	668721.5
Shikoku	7	6	269018.6	227014	156927.5	235391.2	294032
Kyushu	23	8	285891.5	235670.3	219183.5	273979.4	1155087.3
Nation	200	9	277846	272864.6	267159.6	276463.7	996271.9

For V/S, r, $V/(S+n-1)$, $V/(S+1)$, and $r\Theta$, integers above are exactly so.

$$\sum_k r^{(k)} \Theta^{(k)} \approx 7041276.6$$

TABLE 5. Detailed *falsified* data

Hokkaido	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	740678	493786	740679	246893	0	246893	0	0	0	2468929
Percentage	30	20	30	10	0	10	0	0	0	100
Upper	3	2	3	1	0	1	0	0	0	10
Seats	2	2	3	1	0	1	0	0	0	9
Lower	2	2	3	1	0	1	0	0	0	9
Perfect	2.7	1.8	2.7	0.9	0	0.9	0	0	0	9
Estimates	2.95	1.8	2.95	0.65	0	0.65	0	0	0	9
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

Tohoku	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1788484	1532987	510996	255498	255498	0	0	0	0	4343463
Percentage	41.18	35.29	11.76	5.88	5.88	0	0	0	0	100
Upper	8	7	2	1	1	0	0	0	0	19
Seats	6	6	2	1	1	0	0	0	0	16
Lower	6	6	2	1	1	0	0	0	0	16
Perfect	6.59	5.65	1.88	0.94	0.94	0	0	0	0	16
Estimates	7.12	6.03	1.68	0.59	0.59	0	0	0	0	16
θ_j	0.999997	0	1×10^{-6}	7×10^{-7}	7×10^{-7}	0	0	0	0	0.999999

K.Kanto	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2171987	1447992	965328	482664	241332	0	0	0	0	5309303
Percentage	40.91	27.27	18.18	9.09	4.55	0	0	0	0	100
Upper	10	6	4	2	1	0	0	0	0	23
Seats	8	6	4	2	1	0	0	0	0	21
Lower	8	6	4	2	1	0	0	0	0	21
Perfect	8.59	5.73	3.82	1.91	0.95	0	0	0	0	21
Estimates	9.11	5.91	3.77	1.64	0.57	0	0	0	0	21
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

M.Kanto	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1905773	1667552	1191109	714666	238222	0	0	0	0	5717322
Percentage	33.33	29.17	20.83	12.5	4.17	0	0	0	0	100
Upper	8	7	5	3	1	0	0	0	0	24
Seats	7	7	5	3	1	0	0	0	0	23
Lower	7	6	5	3	1	0	0	0	0	22
Perfect	7.67	6.71	4.79	2.88	0.96	0	0	0	0	23
Estimates	8	6.94	4.81	2.69	0.56	0	0	0	0	23
θ_j	0.999997	0	2×10^{-6}	4×10^{-6}	1×10^{-6}	0	0	0	0	1.000004

Tokyo	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1456412	1213677	1213677	728207	242736	0	0	0	0	4854709
Percentage	30	25	25	15	5	0	0	0	0	100
Upper	6	5	5	3	1	0	0	0	0	20
Seats	5	5	5	3	1	0	0	0	0	19
Lower	5	4	4	3	1	0	0	0	0	17
Perfect	5.7	4.75	4.75	2.85	0.95	0	0	0	0	19
Estimates	5.95	4.87	4.87	2.73	0.58	0	0	0	0	19
θ_j	0.999998	0	0	3×10^{-6}	2×10^{-6}	0	0	0	0	1.000004

TABLE 5. Detailed *falsified* data (Continued)

H.Shin.	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1459727	973152	486576	243288	0	0	243288	0	0	3406031
Percentage	42.86	28.57	14.29	7.14	0	0	7.14	0	0	100
Upper	7	4	2	1	0	0	1	0	0	15
Seats	5	4	2	1	0	0	1	0	0	13
Lower	5	4	2	1	0	0	1	0	0	13
Perfect	5.57	3.71	1.86	0.93	0	0	0.93	0	0	13
Estimates	6.14	3.93	1.71	0.61	0	0	0.61	0	0	13
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

Tokai	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2268110	2016099	756037	756037	252013	0	0	0	0	6048296
Percentage	37.5	33.33	12.5	12.5	4.17	0	0	0	0	100
Upper	10	9	3	3	1	0	0	0	0	26
Seats	8	8	3	3	1	0	0	0	0	23
Lower	8	8	3	3	1	0	0	0	0	23
Perfect	8.62	7.67	2.88	2.88	0.96	0	0	0	0	23
Estimates	9.06	8	2.69	2.69	0.56	0	0	0	0	23
θ_j	0.999996	1×10^{-6}	0	0	3×10^{-6}	0	0	0	0	1 (exact)

Kinki	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2691028	2446390	978556	1467834	489278	0	244639	0	0	8317725
Percentage	32.35	29.41	11.76	17.65	5.88	0	2.94	0	0	100
Upper	12	11	4	6	2	0	1	0	0	36
Seats	10	10	4	6	2	0	1	0	0	33
Lower	10	10	4	6	2	0	1	0	0	33
Perfect	10.68	9.71	3.88	5.82	1.94	0	0.97	0	0	33
Estimates	11.15	10.09	3.74	5.85	1.62	0	0.56	0	0	33
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

Chugoku	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	1624692	696297	232099	232099	232099	232099	0	0	0	3249385
Percentage	50	21.43	7.14	7.14	7.14	7.14	0	0	0	100
Upper	8	3	1	1	1	1	0	0	0	15
Seats	6	3	1	1	1	1	0	0	0	13
Lower	6	3	1	1	1	1	0	0	0	13
Perfect	6.5	2.79	0.93	0.93	0.93	0.93	0	0	0	13
Estimates	7.5	2.93	0.64	0.64	0.64	0.64	0	0	0	13
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

Shikoku	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	908059	454030	227015	0	0	227015	0	0	0	1816119
Percentage	50	25	12.5	0	0	12.5	0	0	0	100
Upper	4	2	1	0	0	1	0	0	0	8
Seats	3	2	1	0	0	1	0	0	0	7
Lower	3	2	1	0	0	1	0	0	0	7
Perfect	3.5	1.75	0.88	0	0	0.88	0	0	0	7
Estimates	4	1.75	0.63	0	0	0.63	0	0	0	7
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

TABLE 5. Detailed *falsified* data (Continued)

Kyushu	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	2356709	1649697	471342	471342	471342	0	0	235671	0	5656103
Percentage	41.67	29.17	8.33	8.33	8.33	0	0	4.17	0	100
Upper	11	8	2	2	2	0	0	1	0	26
Seats	9	7	2	2	2	0	0	1	0	23
Lower	9	7	2	2	2	0	0	1	0	23
Perfect	9.58	6.71	1.92	1.92	1.92	0	0	0.96	0	23
Estimates	10.33	7.08	1.67	1.67	1.67	0	0	0.58	0	23
θ_j	0.999996	0	0	0	0	0	0	0	0	0.999996

Total	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	19371659	14591659	7773414	5598528	2422520	706007	487927	235671	0	51187385
Percentage	37.84	28.51	15.19	10.94	4.73	1.38	0.95	0.46	0	100
Upper	87	64	32	23	10	3	2	1	0	222
Seats	69	60	32	23	10	3	2	1	0	200
Lower	69	58	31	23	10	3	2	1	0	197
Perfect	75.69	57.01	30.37	21.87	9.47	2.76	1.91	0.92	0	200
Estimates	81.32	59.33	29.16	19.74	6.78	1.92	1.17	0.58	0	200
$\sum_k \theta_j^{(k)}$	11	1×10^{-6}	3×10^{-6}	8×10^{-6}	7×10^{-6}	0	0	0	0	11
$\sum_k S^{(k)} p_j^{(k)}$	75.7	56.96	30.28	21.95	9.55	2.7	1.9	0.96	0	200

Nation	LDP	NFP	MIN	JCP	SDP	NSP	NPS	JR	DRL	Total
Votes	19371659	14591659	7773414	5598528	2422520	706007	487927	235671	0	51187385
Percentage	37.84	28.51	15.19	10.94	4.73	1.38	0.95	0.46	0	100
Upper	78	59	31	22	9	2	1	0	0	202
Seats	77	58	31	22	9	2	1	0	0	200
Lower	76	57	30	21	9	2	1	0	0	196
Perfect	75.69	57.01	30.37	21.87	9.47	2.76	1.91	0.92	0	200
Estimates	76.7	57.65	30.48	21.81	9.15	2.31	1.44	0.44	0	200
θ_j	0.25	0.19	0	0.33	0.66	0.82	0.95	0.94	0	4.13

TABLE 5. Detailed *falsified* data (Continued)

Hokkaido	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	740678	1234464	1975143	246893	246893
Percentage	30	50	80	10	10
Upper	3	5	8	1	1
Seats	2	4	7	1	1
Lower	2	4	7	1	1
Perfect	2.7	4.5	7.2	0.9	0.9
Estimates	2.95	4.75	7.7	0.65	0.65
θ_G	0.999996	0.999996	0.999996	0	0

Tohoku	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	2043982	3321471	3832467	510996	0
Percentage	47.06	76.47	88.24	11.76	0
Upper	8	14	15	2	0
Seats	7	12	14	2	0
Lower	7	12	14	1	0
Perfect	7.53	12.24	14.12	1.88	0
Estimates	7.71	13.15	14.82	1.18	0
θ_G	0.999997	0.999997	0.999998	1×10^{-6}	0

K.Kanto	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	2413319	3619979	4585307	723996	0
Percentage	45.45	68.18	86.36	13.64	0
Upper	10	16	19	3	0
Seats	9	14	18	3	0
Lower	9	14	18	2	0
Perfect	9.55	14.32	18.14	2.86	0
Estimates	9.68	15.02	18.8	2.2	0
θ_G	0.999996	0.999996	0.999996	0	0

M.Kanto	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	2143995	3573325	4764434	952888	0
Percentage	37.5	62.5	83.33	16.67	0
Upper	9	16	20	4	0
Seats	8	14	19	4	0
Lower	8	14	19	3	0
Perfect	8.62	14.37	19.17	3.83	0
Estimates	8.56	14.94	19.75	3.25	0
θ_G	0.999998	0.999997	0.999999	5×10^{-6}	0

Tokyo	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	1699148	2670089	3883766	970943	0
Percentage	35	55	80	20	0
Upper	7	12	16	4	0
Seats	6	10	15	4	0
Lower	6	10	15	3	0
Perfect	6.65	10.45	15.2	3.8	0
Estimates	6.52	10.82	15.7	3.3	0
θ_G	1.00000082	0.999998	0.999998	6×10^{-6}	0

TABLE 5. Detailed *falsified* data (Continued)

H.Shin.	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	1703015	2432879	2919455	243288	243288
Percentage	50	71.43	85.71	7.14	7.14
Upper	7	11	12	1	1
Seats	6	9	11	1	1
Lower	6	9	11	1	1
Perfect	6.5	9.29	11.14	0.93	0.93
Estimates	6.75	10.07	11.79	0.61	0.61
θ_G	0.999996	0.999996	0.999996	0	0

Tokai	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	2520123	4284209	5040246	1008050	0
Percentage	41.67	70.83	83.33	16.67	0
Upper	10	18	20	4	0
Seats	9	16	19	4	0
Lower	9	16	19	3	0
Perfect	9.58	16.29	19.17	3.83	0
Estimates	9.62	17.06	19.75	3.25	0
θ_G	0.999999	0.999997	0.999997	3×10^{-6}	0

Kinki	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	3424945	5137418	6115974	1957112	244639
Percentage	41.18	61.76	73.53	23.53	2.94
Upper	14	22	26	8	1
Seats	13	20	24	8	1
Lower	12	20	24	7	1
Perfect	13.59	20.38	24.26	7.76	0.97
Estimates	13.32	21.24	24.97	7.47	0.56
θ_G	0.999996	0.999996	0.999996	0	0

Chugoku	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	1856791	2320989	2553088	464198	232099
Percentage	57.14	71.43	78.57	14.29	7.14
Upper	9	12	12	2	1
Seats	7	9	10	2	1
Lower	7	9	10	1	1
Perfect	7.43	9.29	10.21	1.86	0.93
Estimates	8.14	10.43	11.07	1.29	0.64
θ_G	0.999996	0.999996	0.999996	0	0

Shikoku	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	908059	1362089	1589104	0	227015
Percentage	50	75	87.5	0	12.5
Upper	4	6	6	0	1
Seats	3	5	6	0	1
Lower	3	5	6	0	1
Perfect	3.5	5.25	6.12	0	0.88
Estimates	4	5.75	6.37	0	0.63
θ_G	0.999996	0.999996	0.999996	0	0

TABLE 5. Detailed *falsified* data (Continued)

Kyushu	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	2828051	4006406	4477748	942684	235671
Percentage	50	70.83	79.17	16.67	4.17
Upper	13	19	20	4	1
Seats	11	16	18	4	1
Lower	11	16	18	3	1
Perfect	11.5	16.29	18.21	3.83	0.96
Estimates	12	17.42	19.08	3.33	0.58
θ_G	0.999996	0.999996	0.999996	0	0

Total	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	22282106	33963318	41736732	8021048	1429605
Percentage	43.53	66.35	81.54	15.67	2.79
Upper	94	151	174	33	6
Seats	81	129	161	33	6
Lower	80	129	161	25	6
Perfect	87.06	132.7	163.07	31.34	5.59
Estimates	89.27	140.65	169.81	26.53	3.67
$\sum_k \theta_G^{(k)}$	11	11	11	0.00001	0
$\sum_k S^{(k)} p_G^{(k)}$	87.15	132.67	162.94	31.5	5.56

Nation	LDP+NPS+SDP	LDP+NFP	LDP+MIN+NFP	JCP+SDP	DRL+JR+NPS+NSP
Votes	22282106	33963318	41736732	8021048	1429605
Percentage	43.53	66.35	81.54	15.67	2.79
Upper	89	136	167	32	5
Seats	87	135	166	31	3
Lower	86	133	163	30	3
Perfect	87.06	132.7	163.07	31.34	5.59
Estimates	87.3	134.36	164.84	30.97	4.2
θ_G	1.86	0.44	0.44	0.99	2.7

	S	n	V/S	r	$V/(S+n-1)$	$V/(S+1)$	$r\Theta$
Hokkaido	9	5	274325.44	246893	189917.62	246892.9	246892
Tohoku	16	5	271466.44	255497.83	217173.15	255497.82	255497.67
K.Kanto	21	5	252823.95	241332	212372.12	241331.95	241331
M.Kanto	23	5	248579.22	238221.71	211752.67	238221.75	238222.57
Tokyo	19	5	255511	242735.4	211074.3	242735.45	242736.4
H.Shin.	13	5	262002.38	243288	200354.76	243287.93	243287
Tokai	23	5	262969.39	252012.33	224010.96	252012.33	252012.33
Kinki	33	6	252052.27	244639	218887.5	244638.97	244638
Chugoku	13	6	249952.69	232099	180521.39	232098.93	232098
Shikoku	7	4	259445.57	227015	181611.9	227014.88	227014
Kyushu	23	6	245917.52	235671	202003.68	235670.96	235670
Nation	200	8	255936.93	250755.29	247282.05	254663.61	1036326.94

For V/S, r, $V/(S+n-1)$, $V/(S+1)$, and $r\Theta$, integers above are exactly so.

$$\sum_k r^{(k)}\Theta^{(k)} \approx 2659398.97$$

References

Anonymous authors, Election news, *Asahi Evening News*, Newspaper on October 21, 1996a, p. 4.

Anonymous authors, Election news (in Japanese), *Tokyo Shimbun*, Morning newspaper on October 22, 1996b, p. 8.

Anonymous authors, Election news (in Japanese), *Akahata* (for Kita-Kanto district), Daily newspaper on October 22, 1996c, p. 13.

W. Birke, *European Elections by Direct Suffrage: A Comparative Study of the Electoral Systems Used in Western Europe and their Utility for the Direct Election of a European Parliament*, (A. W. Sythoff, Leyden, 1961).

G. W. Cox, Is the single nontransferable vote superproportional? Evidence from Japan and Taiwan, *American Journal of Political Science*, 40 (1996) 740–755.

H. Fujita, *Constituencies in Japan (Nippon no Senkyokusei)* (in Japanese), (Tōyō Keizai Shimpōsha, Tokyo, 1978).

[Forename?] Hagenbach-Bischoff, *Die Frage der Einfürung einer Proportionalvertretung statt des absoluten Mehrs*, ([Publisher?], [Published city?] Switzerland, 1888).
[Reference at second hand from Birke (1961, p. 62)].

[Forename?] Hagenbach-Bischoff, *Verteilung der Sitze bei der Proportionalvertretung (Zeitschrift für die gesamte Staatswissensehft)*, ([Publisher?], [Published city?] Switzerland, 1908).
[Reference at second hand from Moriguchi (1925, p. 619)].

[Forename?] Hagenbach-Bischoff, H. Studer, and R. Pictet, *Berechtigung und Ausführbarkeit der proportionalen Vertretung bei unseren politischen Wahlen.—Ein freies Wort an das Schweizervolk und seine Räthe.* (Schweighauserische Verlagsbuchhandlung, Basel, 1884).

V. d'Hondt, *Question Électorale. La Représentation Proportionelle des Partis*, (Ghent, [Published city?] Belgium, 1878).
[Reference at second hand from Moriguchi (1925, p. 285), Mizuki (1967, p. 264), and Rokkan (1968)].

V. d'Hondt, *Système Practique et Raisonné de Représentation Proportionelle*, (Muquardt, Brussels, 1882).
[Reference at second hand from Moriguchi (1925, p. 286), Mizuki (1967, p. 264), and Rokkan (1968)].

A. Lijphart and R. W. Gibberd, Thresholds and payoffs in list systems of proportional representation, *European Journal of Political Research* 5 (1977) 219–244.

S. Mizuki, *Electoral Systems (Senkyo Seidoron)* (in Japanese), (Yūshindō, Tokyo, 1967).

S. Moriguchi, *Research in Proportional Representation (Hirei Daihyōhō no Kenkyū)* (in Japanese), (Yūhikaku, Tokyo, 1925, reissued in 1990).

S. Nisihira, *Proportional Representation—Suggestion Based on International Comparison (Hirei Daihyōsei—Kokusai Hikaku ni Motozuku Teian)* (in Japanese), (Chūō Kōronsha, Tokyo, 1981).

S. Nisihira, *Electoral Systems Seen by Statistics (Tōkei de Mita Senkyo no Shikumi)* (in Japanese), (Kōdansha, Tokyo, 1990).

D. W. Rae, *The Political Consequence of Electoral Laws*, Revised edn. (Yale Univ. Press, New Haven and London, 1971).

D. Rae, V. Hanby, and J. Loosemore, Thresholds of representation and thresholds of exclusion: An analytic note on electoral systems, *Comparative Political Studies* 3 (1971) 479–488.

S. Rokkan, Elections: Electoral systems, in: *International Encyclopedia of the Social Sciences* vol. 5, (Crowell Collier and Macmillan, New York, 1968) pp. 6–21.

A. Sainte-Laguë, La représentation proportionelle et la methode des moindres carrés, *Academie des Sciences: Compte Rendus Hebdomadaires* 151 (1910) 377–378. [English translation is included in Lijphart and Gibberd (1977, pp. 241–242) above. "Proportional representation and the method of least squares."]

R. Taagepera, and M. S. Shugart, *Seats and Votes*, (Yale Univ. Press, New Haven and London, 1989).

O. Yamamoto, T. Itagaki, T. Miyazaki, Y. Hoshi, A. Okumura, and M. Sasaki, Election news (in Japanese), *Asahi Shimbun*, Morning newspaper on October 22, 1996, p. 11.